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Which infra-nilmanifolds admit an expanding map or an Anosov diffeomorphism?

A study of the dynamical properties of self-maps on infra-nilmanifolds

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor in Science

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Abstract

Expanding maps and Anosov diffeomorphisms are important types of dynamical systems since they were among the first examples with structural stability and chaotic behavior. Every closed manifold admitting an expanding map is homeomorphic to an infra-nilmanifold and it is conjectured that the same is true for manifolds admitting an Anosov diffeomorphism. This motivates the research of expanding maps and Anosov diffeomorphisms on infra-nilmanifolds.

Although, up to homeomorphism, infra-nilmanifolds are the only closed manifolds supporting an expanding map, not every infra-nilmanifold admits an expanding map. Similarly the existence of an Anosov diffeomorphism on an infra-nilmanifold puts strong conditions on its fundamental group. This dissertation studies which infra-nilmanifolds admit an expanding map or an Anosov diffeomorphism. Because of the algebraic nature of infra-nilmanifolds, these questions are translated into studying the group morphisms of their fundamental groups, which are exactly the almost-Bieberbach groups. The main results of this essay give algebraic methods for deciding whether a given infra-nilmanifold admits an expanding map or an Anosov diffeomorphism.

The proofs in this dissertation combine methods from different branches in mathematics, including (geometric) group theory, number theory, Lie algebras, linear algebraic groups and representation theory of finite groups. The first part of this thesis gives the necessary background about the definitions and results in these areas which are needed throughout the following chapters. The emphasis of this first part is on self-maps of infra-nilmanifolds and the relation to expanding maps and Anosov diffeomorphisms.

The second part focuses on the situation of expanding maps. The main result gives an algebraic criterion to decide whether an infra-nilmanifold admits an expanding map or not. This criterion only depends on the covering Lie group and more specific on the existence of a positive grading on the corresponding Lie algebra. The proof of this result consists of two steps which use different methods. The first step deals with group morphisms of commensurable nilpotent groups. These techniques are also useful for determining the periodic points for a big class of self-maps on infra-nilmanifolds. The second step involves gradings on Lie algebras and the relation to automorphisms on these Lie algebras.

The third part of the thesis discusses the existence of Anosov diffeomorphisms on infra-nilmanifolds. Since every Anosov diffeomorphism on an infra-nilmanifold can be lifted to a covering nilmanifold, a first step is to understand which nilmanifolds admit an Anosov diffeomorphism or equivalently to study Anosov Lie algebras. A new method for constructing Anosov Lie algebras is given which answers many open existence questions about these maps. The new examples include Anosov diffeomorphisms of minimal signature, Anosov Lie algebras of minimal type and a nilmanifold admitting an Anosov diffeomorphism but no expanding map. Once we understand which nilmanifolds in a certain class of infra-nilmanifolds admit an Anosov diffeomorphism, the next step is to study the infra-nilmanifolds in this same class. This essay gives an algebraic description of the infra-nilmanifolds modeled on free nilpotent Lie groups which admit an Anosov diffeomorphism.

In the final chapter, we give some directions for future research by stating open questions which originate from this dissertation. This chapter also contains new results connecting this dissertation to other recent research and describes some methods for tackling these problems.

Beknopte samenvatting

Expanderende afbeeldingen en Anosov diffeomorfismes zijn belangrijke voorbeelden van dynamische systemen aangezien ze structureel stabiel en chaotisch zijn. Elke gesloten variëteit die een expanderende afbeelding toelaat is homeomorf met een infra-nilvariëteit en er wordt vermoed dat hetzelfde waar is voor variëteiten met een Anosov diffeomorfisme. Dit motiveert het onderzoek naar expanderende afbeeldingen en Anosov diffeomorfismes op infra-nilvariëteiten.

Alhoewel expanderende afbeeldingen, op homeomorfisme na, enkel op infra-nilvariëteiten bestaan, laat toch niet elke infra-nilvariëteit een expanderende afbeelding toe. Ook het bestaan van een Anosov diffeomorfisme legt sterke voorwaarden op de fundamentaalgroep van de infra-nilvariëteit. Deze verhandeling bestudeert welke infra-nilvariëteiten een expanderende afbeelding of een Anosov diffeomorfisme toelaten. Door de algebraïsche constructie van deze infra-nilvariëteiten zijn deze vragen equivalent met het bestuderen van groepsmorfismes tussen hun fundamentaalgroepen, namelijk de bijna-Bieberbachgroepen. De belangrijkste resultaten van dit onderzoek geven algebraïsche methodes om te beslissen of een gegeven infra-nilvariëteit een expanderende afbeelding of een Anosov diffeomorfisme toelaat.

De bewijzen in deze verhandeling combineren technieken uit verschillende takken binnen de wiskunde waaronder (geometrische) groepentheorie, getaltheorie, Lie algebras, lineaire algebraïsche groepen en representatietheorie van eindige groepen. Het eerste deel van deze doctoraatsthesis geeft de noodzakelijke achtergrond over de definities en resultaten in deze gebieden voor de volgende hoofdstukken. De nadruk van dit eerste deel ligt op zelfafbeeldingen van infra-nilvariëteiten en het verband met expanderende afbeeldingen en Anosov diffeomorfismes.

Het tweede deel bekijkt de situatie van expanderende afbeeldingen. Het belangrijkste resultaat is een algebraïsch criterium om te beslissen of een infra-nilvariëteit al dan niet een expanderende afbeelding toelaat. Dit criterium

hangt enkel af van de overdekkende Lie groep en meer specifiek van het bestaan van een positieve gradering op de bijhorende Lie algebra. Het bewijs van dit resultaat bestaat uit twee stappen die elk verschillende technieken gebruiken. De eerste stap bepaalt het verband tussen groepsmorfismes van commensurabele nilpotente groepen. Deze technieken zijn ook bruikbaar om de periodische punten van een grote klasse van zelfafbeeldingen op infra-nilvariëteiten te bepalen. De tweede stap gebruikt graderingen op Lie algebras en het verband met automorfismes op deze Lie algebras.

Het derde deel van deze doctoraatsthesis behandelt het bestaan van Anosov diffeomorfismes op infra-nilvariëteiten. Elk Anosov diffeomorfisme kan gelift worden naar de overdekkende nilvariëteit, daarom is de eerste stap begrijpen welke nilvariëteiten een Anosov diffeomorfisme toelaten. Dit laatste is equivalent met het bestuderen van Anosov Lie algebras. Deze thesis geeft een nieuwe methode voor het construeren van Anosov Lie algebras die veel van de open vragen over het bestaan van deze afbeeldingen beantwoordt. De nieuwe voorbeelden van Anosov Lie algebras omvatten onder andere Anosov diffeomorfismen van minimale signatuur, Anosov Lie algebras van minimaal type en een nilvariëteit die een Anosov diffeomorfisme toelaat maar geen expanderende afbeelding. Eens we begrijpen welke nilvariëteiten binnen een bepaalde klasse een Anosov diffeomorfisme toelaten is de volgende stap om de infra-nilvariëteiten binnen dezelfde klasse te bekijken. Deze verhandeling geeft een algebraïsche beschrijving van de infra-nilvariëteiten gemodelleerd op vrije nilpotente Lie groep die een Anosov diffeomorfisme toelaten.

In het laatste hoofdstuk geven we enkele richtingen aan voor verder onderzoek aan de hand van open vragen die resulteren uit deze verhandeling. Dit hoofdstuk bevat ook enkele nieuwe resultaten die deze thesis verbinden met ander recent onderzoek en beschrijft enkele methodes om deze problemen aan te pakken.

Contents

Chapter 1

Introduction

This dissertation studies the existence of expanding maps and Anosov diffeomorphisms on infra-nilmanifolds. To motivate the importance of these dynamical systems, we start with the typical examples which one should keep in mind when studying these types of maps.

Example 1.1. Consider the circle S^1 as the unit circle in the complex plane,

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

We consider S^1 as a Riemannian manifold for the Riemannian metric it inherits as a submanifold of the complex plane $\mathbb{C}$.

Let $n \in \mathbb{Z}$ be any integer with $|n| > 1$ and consider the differentiable map $f_n : S^1 \to S^1$ defined as

$$f_n(z) = z^n$$

for every $z \in S^1$. The map f_n forms a self-cover of the circle S^1. Together with the universal covering map $p : \mathbb{R} \to S^1$ given by

$$p(x) = e^{2\pi i x}$$

for every $x \in \mathbb{R}$ these form all the non-trivial covering maps of S^1 up to isomorphism of covering maps.

First we describe the periodic points of f_n. The manifold S^1 forms an abelian group and let $T \leq S^1$ be the torsion subgroup of S^1. The group T consists exactly of the roots of unity in $\mathbb{C}$. Consider the subset

$$T_n = \{z \in T \mid \gcd(\mathrm{ord}(z), n) = 1\} \subseteq T$$

of elements for which the order is coprime to n. The set T_n is exactly the set of periodic points of f_n and note that T_n is a dense subset of S^1.

For every open $U \subseteq S^1$, there exists a $k > 0$ such that $f_n^k(U) = S^1$. This implies that for all open subsets $U, V \subseteq S^1$, there exists $k_0 \in \mathbb{N}$ such that

$$f_n^k(U) \cap V \neq \varnothing$$

for all $k \geq k_0$, a property we call topologically mixing. Another consequence is that for every $0 < \epsilon < \pi$ and $\delta > 0$ and $x \in S^1$, there exists $y \in S^1$ with $d(x, y) < \delta$ but

$$d(f_n^k(x), f_n^k(y)) > \epsilon$$

for some $k > 0$, which we call sensitive dependence on initial conditions. A map satisfying these two properties and having dense periodic points is called chaotic. In particular, the map f_n is chaotic for all $|n| > 1$.

The derivative Df_n satisfies the property

$$\|Df_n^k(v)\| = |n|^k \|v\| \tag{1.1}$$

and since $|n| > 1$, this implies that f_n expands the length of every tangent vector. Small deformations of f_n satisfy the same property and thus their dynamical properties are identical. If every small deformation has exactly the same dynamical properties as the original map, we call the map structurally stable. The map f_n is thus structurally stable.

The class of expanding maps generalizes this example to a bigger class of closed manifolds. The defining property is exactly that tangent vectors are expanded by the map as given in Equation (1.1), see Definition 3.16.

The examples of Anosov diffeomorphisms are slightly more complicated, although they have similar properties.

Example 1.2. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

which has 2 distinct eigenvalues λ and λ^{-1} satisfying $\lambda > 1 > \lambda^{-1} > 0$. We consider the 2-torus $\mathbb{T}^2$ as the quotient $\mathbb{Z}^2 \backslash \mathbb{R}^2$. Since $A(\mathbb{Z}^2) = \mathbb{Z}^2$, the matrix A induces a diffeomorphism on the 2-torus $\mathbb{T}^2$, which we denote by

$$f_A : \mathbb{T}^2 \to \mathbb{T}^2.$$

Consider for every $n > 0$ the set

$$X_n = \left\{ \left(\frac{k_1}{n}, \frac{k_2}{n} \right) \;\middle|\; k_i \in \mathbb{Z} \right\} \subseteq \mathbb{Q}^2.$$

The set X_n is mapped to itself under A and therefore also the finite set $p(X_n)$ is invariant under f_A. Since f_A is a bijection, this implies that every point of $p(X_n)$ is a periodic point of f_A and thus the set $p(\mathbb{Q}^2)$ is exactly the set of periodic points of f_A. Again the periodic pints form a dense subset of the manifold, just as in Example 1.1.

The map f_A preserves volume and every open subset $U \subseteq \mathbb{T}^2$ has a non-zero volume. This implies that for all open subsets $U, V \subseteq \mathbb{T}^2$, there exists some $k > 0$ such that

$$f^k(U) \cap V \neq \varnothing,$$

which is similar to the property of Example 1.1. In fact the map f_A is also topological mixing.

Consider the eigenvector $v_{\lambda^{-1}}$ corresponding to the eigenvalue λ^{-1} and assume that $\|v_{\lambda^{-1}}\| = 1$. For every $\mathbb{Z}^2 + x \in \mathbb{T}^2$, we have that

$$y_n = f_A^n(\mathbb{Z}^2 + x + \epsilon v_{\lambda^{-1}}) = \mathbb{Z}^2 + A^n(x) + \epsilon\lambda^{-n} v_{\lambda^{-1}}$$

and for n sufficiently big, the distance $d(y_n, x)$ is large relative to the distance $d(y_0, x) = \epsilon$. Again these observations imply that the map f_A is chaotic.

Let V_λ and $V_{\lambda^{-1}}$ be the eigenspaces corresponding to the eigenvalues λ and λ^{-1}. The direct sum $\mathbb{R}^2 = V_\lambda \oplus V_{\lambda^{-1}}$ satisfies the following property:

$$\begin{aligned} &\forall v \in V_\lambda,\ \forall n \in \mathbb{N} : \|A^n(v)\| = \lambda^n \|v\|, \\ &\forall v \in V_{\lambda^{-1}},\ \forall n \in \mathbb{N} : \|A^n(v)\| = \lambda^{-n} \|v\| \end{aligned} \tag{1.2}$$

where $\|\ \|$ is the Euclidean norm on $\mathbb{R}^2$. The linear map A is equal to the derivative of f_A in every point of $\mathbb{T}^2$. Note that small deformations of the map f_A satisfy a similar property as in Equation (1.2) around every point. Again, this implies that the map f_A is structurally stable.

Anosov diffeomorphism can be defined on every closed manifold by generalizing Equation (1.2), see Definition 3.23. Because Anosov diffeomorphisms combine both an expanding and a contracting direction, their properties are harder to study.

An important research question is to investigate which manifolds admit these interesting types of dynamical systems.

Research question 1. Which closed Riemannian manifolds admit an expanding map or an Anosov diffeomorphism?

Examples 1.1 and 1.2 show that S^1 admits an expanding map and $\mathbb{T}^2 \approx S^1 \times S^1$ admits an Anosov diffeomorphism. Both manifolds S^1 and $\mathbb{T}^2$ are examples of flat manifolds. It is easy to construct other examples on the tori $\mathbb{T}^n$ of higher dimension.

In [99] the first example was given of an Anosov diffeomorphism on a manifold which is not flat. This example of S. Smale is on an infra-nilmanifold of dimension 6, where infra-nilmanifolds form a class of closed manifolds which naturally generalize the flat manifolds. Chapter 2 introduces these manifolds in detail.

M. Gromov showed that the infra-nilmanifolds form up to homeomorphism the only class of closed manifolds admitting an expanding map.

Theorem 1.3 (Gromov)**.** *Every closed manifold admitting an expanding map is homeomorphic to an infra-nilmanifold.*

For Anosov diffeomorphisms, the same result is believed to be true.

Conjecture 1.4. Every closed manifold admitting an Anosov diffeomorphism is homeomorphic to an infra-nilmanifold.

This motivates the study of Research Question 1 for the situation of infra-nilmanifolds.

Research question 2. Which infra-nilmanifolds admit an expanding map or an Anosov diffeomorphisms?

This dissertation gives methods for answering Research question 2.

The techniques used to study this question also answer some other related questions, like the existence of non-trivial self-covers on infra-nilmanifolds or the study of periodic points of self-maps on infra-nilmanifolds.

Overview of the main results

We end this introduction by giving an overview of the main results in this thesis, as well as references to the original papers describing these results.

The first main result gives a complete algebraic characterization of the infra-nilmanifolds admitting an expanding map which only depends on the covering Lie group of the infra-nilmanifold.

Main theorem 1. *Let $\Gamma\backslash G$ be an infra-nilmanifold modeled on the Lie group G with corresponding Lie algebra $\mathfrak{g}$. Then the following statements are equivalent.*

(1) *The infra-nilmanifold* $\Gamma\backslash G$ *admits an expanding map.*

(2) *The Lie algebra* $\mathfrak{g}$ *has a positive grading.*

(3) *The Lie group* G *has an expanding automorphism.*

The same techniques give us a criterion for the existence of a non-trivial self-cover, i.e. a self-cover which is not a homeomorphism.

Main theorem 2. *Let* $\Gamma\backslash G$ *be an infra-nilmanifold modeled on a Lie group* G *with corresponding Lie algebra* $\mathfrak{g}$*. Then the following statements are equivalent.*

(1) *The infra-nilmanifold* $\Gamma\backslash G$ *admits a non-trivial self-cover.*

(2) *The group* Γ *is not cohopfian.*

(3) *The Lie algebra* $\mathfrak{g}$ *has a non-trivial non-negative grading.*

(4) *The Lie group* G *has a partially expanding automorphism.*

Both theorems are described in the paper [42], which is heavily based on the results of [32].

The techniques we develop for expanding maps also give us information about periodic points and eventually periodic points of more general maps on infra-nilmanifolds. The set of periodic points forms a dense subset if it is non-empty.

Main theorem 3. *The set of periodic points of an affine infra-nilmanifold endomorphism is empty or dense in the infra-nilmanifold.*

For eventually periodic points, we have a complete description of the set of these points.

Main theorem 4. *Let* $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ *be an affine infra-nilmanifold endomorphism induced by the affine transformation* $\alpha = (g, \delta) \in \mathrm{Aff}(G)$*. Denote by* $p : G \to \Gamma\backslash G$ *the projection map and by* $H^{\mathbb{R}}$ *the subgroup of* G *defined as*

$$H^{\mathbb{R}} = \{h \in G \mid \exists k > 0 : \delta^k(h) = h\}.$$

If $\bar{\alpha}$ *has a periodic point* Γg_0*, then*

$$\mathrm{ePer}(\bar{\alpha}) = p(N^{\mathbb{Q}} g_0 H^{\mathbb{R}}).$$

These results are written down in the paper [43].

The next part studies the situation of Anosov diffeomorphism. The first step is to understand the nilmanifolds which admit an Anosov diffeomorphism or equivalently to describe the rational Lie algebras which are Anosov. Our main result, see [41], is rather technical but is very helpful in constructing new examples.

Main theorem 5. *Let* $\mathfrak{n}^{\mathbb{Q}}$ *be a rational Lie algebra and* $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ *a representation. Suppose there exists a Lie algebra automorphism* $f : \mathfrak{n}^E \to \mathfrak{n}^E$ *such that* $\rho_\sigma f \rho_{\sigma^{-1}} = f^\sigma$ *for all* $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$*. Then there also exists a rational form* $\mathfrak{m}^{\mathbb{Q}} \subseteq \mathfrak{n}^E$ *such that* f *induces an automorphism of* $\mathfrak{m}^{\mathbb{Q}}$*.*

If all eigenvalues of f *are algebraic units of absolute value different from* 1*, then* $\mathfrak{m}^{\mathbb{Q}}$ *is Anosov.*

Finally, we also have a criterion for deciding whether an infra-nilmanifold modeled on a free nilpotent Lie group admits an Anosov diffeomorphism.

Main theorem 6. *Let* $\Gamma \backslash G$ *be an infra-nilmanifold modeled on a free c-step nilpotent Lie group* G *and consider the associated abelianized rational holonomy representation* $\overline{\varphi} : F \to \mathrm{Aut}\left({}^{N_{\mathbb{Q}}}\!\big/_{[N_{\mathbb{Q}}, N_{\mathbb{Q}}]} \right)$ *where* F *is the holonomy group of* Γ*. Then the following statements are equivalent:*

$$\Gamma \backslash G \textit{ admits an Anosov diffeomorphism.}$$
$$\Updownarrow$$
$$\textit{Every } \mathbb{Q}\textit{-irreducible component of } \bar{\varphi} \textit{ that occurs with multiplicity } m\textit{,}$$
$$\textit{splits in more than } \tfrac{c}{m} \textit{ components when seen as a representation over } \mathbb{R}.$$

This result is published in [31].

The proof of these main results combines techniques from different areas in mathematics, including Lie algebras, number theory, representation theory of finite groups and linear algebraic groups. The first part of this thesis gives the necessary background about these different subjects. Part II then discusses the theorems about expanding maps, non-trivial self-covers and periodic points of self-maps on infra-nilmanifolds. In Part III of this thesis we restrict the attention to Anosov diffeomorphisms as given in the last two results.

Part I

Background

Chapter 2

Self-maps on infra-nilmanifolds

This chapter gives an introduction to self-maps on infra-nilmanifolds, which are the natural generalization of flat manifolds, and also fixes some notations for the thesis. Every infra-nilmanifold is a closed manifold with a contractible universal cover and a fundamental group which is virtually nilpotent. Therefore the self-maps of these manifolds can be studied in an algebraic way.

In the first section we describe the relation between Lie groups and Lie algebras. Next we recall some elementary properties of covering spaces to introduce infra-nilmanifolds. The structure of flat manifolds is well known because of the Bieberbach theorems and we show how these results can be generalized to infra-nilmanifolds in the fourth section. A general class of self-maps on infra-nilmanifolds is also introduced in this chapter, namely the affine infra-nilmanifold endomorphism. Finally, we introduce the rational holonomy representation of infra-nilmanifolds, which plays an important role in this thesis.

During this chapter and the thesis, we always assume that the manifolds and differentiable maps between them are C^∞. If $f : M \to M$ is a differentiable map, then its derivative is denoted as $Df : TM \to TM$.

2.1 Lie algebras and Lie groups

We start with recalling the basic notions about Lie algebras, Lie groups and their relation. The emphasis is on nilpotent Lie algebras and Lie groups, because of their importance for infra-nilmanifolds. A more detailed introduction can be found in [46].

Lie algebra

Definition 2.1. A **Lie algebra** is a vector space $\mathfrak{g}$ over some field E together with a binary operator

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the **Lie bracket**, satisfying the following three properties.

(1) The Lie bracket is bilinear, so for all $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}$ and every $\lambda, \mu \in E$, it holds that

$$[\lambda X_1 + \mu X_2, Y] = \lambda[X_1, Y_1] + \mu[X_2, Y_1]$$

$$[X_1, \lambda Y_1 + \mu Y_2] = \lambda[X_1, Y_1] + \mu[X_1, Y_2].$$

(2) The Lie bracket is alternating, meaning that

$$[X, X] = 0 \ \ \forall X \in \mathfrak{g}.$$

(3) The Lie bracket satisfies the Jacobi identity, i.e.

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \ \ \forall X, Y, Z \in \mathfrak{g}.$$

Since the Lie bracket is bilinear and alternating, it is also anticommutative, meaning that $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$. Lie algebras will always be denoted by gothic letter through this thesis, e.g. $\mathfrak{g}, \mathfrak{h}, \mathfrak{n}$. We only consider finite dimensional Lie algebras in this thesis, where the dimension of a Lie algebra is its dimension as a vector space over E.

Example 2.2. Let A be an algebra over the field E, then A is a Lie algebra for the Lie bracket

$$[a, b] = ab - ba.$$

For example, every matrix subalgebra of $E^{n\times n}$ is a Lie algebra in this way.

A Lie subalgebra $\mathfrak{h}$ is a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\mathfrak{h}$ is closed under the Lie bracket. If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two subalgebras of $\mathfrak{g}$, then the subalgebra $[\mathfrak{h}_1, \mathfrak{h}_2]$ is defined as the smallest subalgebra containing $[X, Y]$ for every $X \in \mathfrak{h}_1$ and $Y \in \mathfrak{h}_2$. An ideal of $\mathfrak{g}$ is a subalgebra I such that $[\mathfrak{g}, I] \subseteq I$. If I is an ideal of the Lie algebra $\mathfrak{g}$, we can define the quotient Lie algebra $\mathfrak{g}/I$ where the Lie bracket between $X + I$ and $Y + I$ is given by

$$[X + I, Y + I] = [X, Y] + I.$$

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras, then a Lie algebra morphism is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ which also preserves the Lie bracket, meaning that

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$$

for all $X, Y \in \mathfrak{g}$. A Lie algebra isomorphism is a Lie algebra morphism which is also invertible. The set of Lie algebra automorphisms of a Lie algebra $\mathfrak{g}$ form a group for the composition and this group is denoted as $\mathrm{Aut}(\mathfrak{g})$.

The lower central series $\gamma_i(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined inductively by $\gamma_1(\mathfrak{g}) = \mathfrak{g}$ and the formula

$$\gamma_{i+1}(\mathfrak{g}) = [\mathfrak{g}, \gamma_i(\mathfrak{g})].$$

Every $\gamma_i(\mathfrak{g})$ is an ideal of the Lie algebra $\mathfrak{g}$ and from the Jacobi identity it follows that $[\gamma_i(\mathfrak{g}), \gamma_j(\mathfrak{g})] \subseteq \gamma_{i+j}(\mathfrak{g})$. We call a Lie algebra $\mathfrak{g}$ nilpotent if there exists some c such that $\gamma_{c+1}(\mathfrak{g}) = 0$ The smallest possible c such that $\gamma_{c+1}(\mathfrak{g}) = 0$ is the nilpotency class of the Lie algebra $\mathfrak{g}$ and we say that $\mathfrak{g}$ is c-step nilpotent.

An abelian Lie algebra is a nilpotent Lie algebra of nilpotency class 1 or equivalently a Lie algebra where the Lie bracket $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. The center $Z(\mathfrak{g})$ of a Lie algebra is defined as

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \forall Y \in \mathfrak{g}\}.$$

Example 2.3. Let A be the algebra of upper triangular matrices over E with 0 on the diagonal. Then A is a nilpotent Lie algebra for the bracket defined in Example 2.2.

Let $d : \mathfrak{g} \to \mathfrak{g}$ be a linear map on the Lie algebra $\mathfrak{g}$. We call d a derivation of the Lie algebra $\mathfrak{g}$ if for every $X, Y \in \mathfrak{g}$, we have that

$$d([X, Y]) = [d(X), Y] + [X, d(Y)].$$

The derivations of a Lie algebra form a Lie algebra for the Lie bracket defined as

$$[d_1, d_2](X) = d_1(d_2(X)) - d_2(d_1(X)).$$

During the remaining part of this dissertation, we will use a superscript ^{E} in the notation of a Lie algebra if we want to emphasize over which field E the Lie algebra is defined. This notation will be useful in the situations where we consider Lie algebras over field extensions of E. If $F \supseteq E$ is a field extension and $\mathfrak{n}^E$ a Lie algebra over E, then

$$\mathfrak{n}^F = \mathfrak{n}^E \otimes_E F$$

forms a Lie algebra by linearly extending the Lie bracket. For example, a rational Lie algebra will be written as $\mathfrak{n}^\mathbb{Q}$ and its complexification $\mathfrak{n}^\mathbb{Q} \otimes \mathbb{C}$ is denoted as $\mathfrak{n}^\mathbb{C}$. A similar notation will be used for the extension of Lie algebra morphisms $\varphi : \mathfrak{g}^E \to \mathfrak{h}^E$, i.e. $\varphi^F : \mathfrak{g}^F \to \mathfrak{h}^F$ denotes the unique extension of φ^F.

Lie group

Definition 2.4. A **Lie group** G is a differentiable manifold such that the multiplication map

$$G \times G \to G : (g, h) \mapsto gh$$

and the inversion map

$$G \to G : g \mapsto g^{-1}$$

are differentiable maps between the manifolds.

A Lie subgroup H of a Lie group is a subgroup $H \leq G$ such that the inclusion map $H \hookrightarrow G$ is an immersion. Let G and H be Lie groups, then a Lie group morphism between G and H is a group morphism $f : G \to H$ such that f is also continuous. In this case, the map $f : G \to H$ is automatically differentiable.

Every group element $g \in G$ induces a diffeomorphism of the Lie group given by left multiplication, i.e. the map

$$L_g : G \to G$$
$$h \mapsto gh$$

is a diffeomorphism of the Lie group G.

Denote by $C^\infty(G)$ the set of differentiable functions $G \to \mathbb{R}$. The set $C^\infty(G)$ is a vector space for the natural operations inherited from the vector space $\mathbb{R}$. In fact, $C^\infty(G)$ forms an algebra where the multiplication $h_1h_2 : G \to \mathbb{R}$ between $h_1, h_2 \in C^\infty(G)$ is given by

$$h_1h_2(g) = h_1(g)h_2(g).$$

A vector field on G is a linear map $X : C^\infty(G) \to C^\infty(G)$ such that

$$X(h_1h_2) = X(h_1)h_2 + h_1X(h_2),$$

i.e. X is a derivation of the algebra $C^\infty(G)$ where derivations on general algebras are defined in the same way as for Lie algebras. Write $\mathfrak{X}(G)$ for the set of all vector fields on G which forms a vector space for the natural operations.

The vector fields $\mathfrak{X}(G)$ are naturally isomorphic to the differentiable sections of the tangent bundle $TG \to G$. Under this isomorphism, a section $Y : G \to TG$ corresponds to the vector field which maps $h \in C^\infty(G)$ to the function

$$g \mapsto D_{Y(g)}(h) \in C^\infty(G)$$

where $D_{Y(g)}$ is the direction derivative in the direction of $Y(g)$.

If $f : G \to H$ is a diffeomorphism, then there exists a linear map $Df : \mathfrak{X}(G) \to \mathfrak{X}(H)$ which maps the vector field $X \in \mathfrak{X}(G)$ to

$$Df(X)(h) = X(h \circ f) \circ f^{-1}$$

where $h \in C^\infty(H)$ and thus $h \circ f \in C^\infty(G)$. A general differentiable map does not induce such a linear map though.

There is a natural Lie bracket on the vector space $\mathfrak{X}(G)$, defined as

$$[X, Y](h) = X(Y(h)) - Y(X(h))$$

for every $X, Y \in \mathfrak{X}(G)$ and $h \in C^\infty(G)$. If $f : G \to H$ is a diffeomorphism, then $Df : \mathfrak{X}(G) \to \mathfrak{X}(H)$ is a Lie algebra morphism, since

$$\begin{aligned} Df([X, Y])(h) &= X(Y(h \circ f)) \circ f^{-1} - Y(X(h \circ f)) \circ f^{-1} \\ &= [Df(X), Df(Y)](h). \end{aligned}$$

Again, this formula does not make sense for general differentiable maps f.

The definition of vector fields and the Lie bracket do not use the structure of G as a Lie group and can in fact be defined for every manifold. But the Lie bracket on $\mathfrak{X}(G)$ does get particularly interesting when working on Lie groups G. We explain this relation between Lie groups and Lie algebras in the next paragraphs.

Relation Lie algebras and Lie groups

The Lie bracket on vector fields is defined for general manifolds M. But the space of vector fields is in general not finite dimensional and differentiable maps do not induce Lie algebra morphisms in a natural way. When restricting to Lie groups G, these problems can be overcome by restricting to left-invariant vector fields.

A vector field $X \in \mathfrak{X}(G)$ is called left invariant if $DL_g(X) = X$ for every $g \in G$. The subspace of left-invariant vector fields forms a Lie subalgebra since

$$DL_g([X, Y]) = [DL_g(X), DL_g(Y)] = [X, Y]$$

where we use that L_g is a diffeomorphism. This subalgebra of left invariant vector fields is denoted as $\mathfrak{g}$. The left invariant vector fields correspond to sections $Y : G \to TG$ with

$$Y(g) = DL_g(Y(e)).$$

The map which associates to every vector field X the tangent vector $Y(e)$ of the corresponding section $Y : G \to TG$ forms an isomorphism between $\mathfrak{g}$ and the tangent space T_eG. In particular, $\mathfrak{g}$ is a finite dimensional real Lie algebra and we call this the Lie algebra corresponding to the Lie group G.

Let $f : G \to H$ be a Lie group morphism and consider the linear map $Df : T_eG \to T_eH$. Since the vector spaces T_eG and T_eH are isomorphic to the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ corresponding to G and H, there is a corresponding linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$. It is an important result that φ is a Lie algebra morphism. Moreover, if G is connected, then the Lie algebra φ uniquely determines the Lie group morphism f. In general, not every Lie algebra morphism is induced by a Lie group morphism.

Example 2.5. Consider the Lie groups $G = S^1$ and $H = \mathbb{R}$. Since these manifolds are 1-dimensional, the Lie algebra $\mathfrak{g}$ corresponding to the both Lie groups is the unique 1-dimensional abelian Lie algebra. There is only Lie group morphism $G \to H$, namely the trivial one.

For every vector $X \in \mathfrak{g} \approx T_eG$, there is a unique Lie group morphism $\varphi_X : \mathbb{R} \to G$ such that $D\varphi_X(1) = X$. The exponential map $\exp : \mathfrak{g} \to G$ is defined as

$$\exp(X) = \varphi_X(1).$$

The Baker-Campbell-Hausdorff formula gives an exact relation between the Lie bracket on $\mathfrak{g}$ and the multiplication on G under the exponential map.

Theorem 2.6 (Baker-Campbell-Hausdorff formula). *Let G be a simply connected and connected Lie group with corresponding Lie algebra $\mathfrak{g}$. Take vectors $X, Y \in \mathfrak{g}$, then*

$$\exp(X)\exp(Y) = \exp\left(X + Y + \sum_{k=2}^{\infty} q_k(X,Y)\right)$$

where the $q_k(X,Y)$ are a rational linear combination of k-fold Lie brackets in X, Y.

There is an explicit formula for the rational combinations $q_m(X,Y)$, for example

$$q_2(X,Y) = \frac{1}{2}[X,Y]$$

and

$$q_3(X,Y) = \frac{1}{12}\big([X,[X,Y]] + [Y,[Y,X]]\big).$$

For simply connected and connected Lie groups G, there is a one-to-one correspondence between group morphisms and Lie algebra morphisms.

Theorem 2.7. *Let G, H be simply connected and connected Lie groups with corresponding Lie algebra $\mathfrak{g}, \mathfrak{h}$, then the map $f \mapsto Df$ induces a bijection between the Lie group morphisms $f : G \to H$ and the Lie algebra morphisms $\varphi : \mathfrak{g} \to \mathfrak{h}$.*

The inclusion of a Lie subgroup corresponds to the inclusion of a Lie subalgebra and in particular, there is also a one-to-one correspondence between Lie subgroups and Lie subalgebras. Under this correspondence, normal subgroups correspond to ideals.

2.2 Covering spaces

Infra-nilmanifolds are a class of closed manifold which have a simply connected and connected nilpotent Lie group as universal cover. Therefore we recall the basic notions of covering spaces and the universal cover of manifolds. The emphasis is on the relation between covering spaces and fundamental groups and we assume that the reader is familiar with fundamental groups of topological spaces as introduced for example in [86]. In this section, we will always assume that E and B are connected and locally path-connected topological spaces.

Covering spaces

Let $p : E \to B$ be a continuous surjection, then we say that an open subset $U \subseteq B$ is evenly covered by p if

$$p^{-1}(U) = \bigcup_{i \in I} U_i$$

is the disjoint union of open subsets U_i such that the restriction $p|_{U_i}$ is a homeomorphism between U_i and U. If every point of B has an open neighborhood which is evenly covered by p, then we call the map p a covering map. We say that $p : E \to B$ is a finite covering map if the inverse image $p^{-1}(b)$ is finite for one $b \in B$ and hence for every $b \in B$.

Let $p : E \to B$ be a covering map and $g : B \to B$ a continuous map. A continuous map $f : E \to E$ is a lift of g if f makes the following diagram commute.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"] & E \arrow[d, "p"] \\ B \arrow[r, "g"] & B \end{tikzcd}$$

If f_1, f_2 are two lifts of the same map $g : B \to B$ such that $f_1(e) = f_2(e)$ for some $e \in E$, then $f_1 = f_2$.

Let $p : E \to B$ be a covering map. A homeomorphism $h : E \to E$ such that $p \circ h = p$ is called a covering transformation or deck transformation of the covering map p. Note that a covering transformation is a lift of the map $\mathbb{1}_B$ and therefore is uniquely determined by its value in one point. The set of all covering transformations forms a group for the composition of maps and we denote this group as $\mathcal{C}(E, p, B)$. The covering map p is called regular if for every $e, e' \in E$ such that $p(e) = p(e')$, there exists a $h \in \mathcal{C}(E, p, B)$ such that $h(e) = e'$. If f is a lift of g, then every other lift of g is of the form $h \circ f$ with $h \in \mathcal{C}(E, p, B)$. This shows that lifts of a map g are not unique.

Let $\Gamma \curvearrowright X$ act continuously on the topological space X, where we denote the action of γ on an element $x \in X$ as ${}^{\gamma}x$. We say that Γ acts freely if for every $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X$ it holds that ${}^{\gamma_1}x = {}^{\gamma_2}x$ implies $\gamma_1 = \gamma_2$. The action is called properly discontinuously if for every $x \in X$, there exists an open $U \subseteq X$ containing x such that the set

$$\{\gamma \in \Gamma \mid {}^{\gamma}U \cap U \neq \varnothing\}$$

is finite. If G acts by isometries on a metric space X, the action is properly continuously if and only if

$$\{\gamma \in \Gamma \mid {}^{\gamma}K \cap K \neq \varnothing\}$$

is finite for every compact subset $K \subseteq X$.

Assume that $p : E \to B$ is a regular covering map and let $\Gamma = \mathcal{C}(E, p, B)$ be the group of covering transformations. The group Γ acts properly discontinuously on E and the quotient space $\Gamma\backslash E$ is naturally isomorphic to the space B. In particular this result holds for the universal cover, which is the covering space $p : E \to B$ such that E is simply connected, i.e. the fundamental group $\pi_1(E)$ is trivial.

A space B has a universal cover if and only if it is semi-locally simply connected and from now on we will assume that our spaces satisfy this property. If M is a manifold, then the universal cover $\bar{M}$ always exists and the universal covering map and every deck transformation are then differentiable maps.

A manifold M with fundamental group π and such that $\bar{M}$ is contractible is called a $K(\pi, 1)$ space. These spaces play an important role, since their continuous maps can be studied as group morphisms on the fundamental group. Denote by $\mathrm{Hom}(G, H)$ the group morphisms between two groups G, H. If X, Y are topological spaces and $x_0 \in X, y_0 \in Y$, then we write $[X, x_0; Y, y_0]$ for the set of homotopy equivalence classes of maps $f : X \to Y$ such that $f(x_0) = y_0$, where we assume that the homotopy fixes the point x_0.

Theorem 2.8. *Let K be a connected CW-complex and M a manifold which is a $K(\pi, 1)$ space. The map*

$$[K, k_0; M, m_0] \to \mathrm{Hom}\left(\pi_1(K, k_0), \pi_1(M, m_0)\right)$$

$$f \mapsto f_*$$

is a bijection.

A proof of this statement is given in [104].

Fundamental group

We discuss the relation between the fundamental group of a topological space and covering maps.

We first show how the group of covering transformations of the universal cover is isomorphic to the fundamental group. Let B be a topological space which has a universal cover $p : \bar{B} \to B$. Fix a point $b \in B$ and $b_0 \in \bar{B}$ such that $p(b_0) = b$. Let Γ be the group of deck transformations of the universal cover $p : \bar{B} \to B$. Take $\gamma \in \Gamma$ and consider a path $f : I \to \bar{B}$ from b_0 to ${}^{\gamma}b_0$. This path projects to a loop $p \circ f$ starting in b and thus to an element of the fundamental group $[p \circ f]_h \in \pi_1(B, b)$.

Since $\bar{B}$ is simply connected, every two paths from b_0 to ${}^{\gamma}b_0$ are homotopic. This implies that the element $[p \circ f]_h$ does not depend on the choice of the path f. So we have a well-defined map

$$\varphi_{b,b_0} : \Gamma \to \pi_1(B, b)$$

and it is a general fact that this map is a group morphism which forms an isomorphism between the group Γ and $\pi_1(B, b)$. In this thesis, we will always identify the fundamental group with the group Γ of covering transformations.

Let $\tilde{p} : E \to B$ be another covering map of B. Since $\bar{B}$ is the universal cover, there exists a covering map $q : \bar{B} \to B$ such that $\tilde{p} \circ q = p$. This shows that the group of covering transformations $\Gamma' = \mathcal{C}(\bar{B}, q, E)$ forms a subgroup of the group Γ. This corresponds to the fact that the covering transformation $\tilde{p}$ induces an injective group morphism on the level of the fundamental groups. The covering map $\tilde{p}$ is regular if and only if Γ' is a normal subgroup of Γ. In this case, the covering transformations of $\tilde{p}$ are isomorphic to ${}^{\Gamma}\!/_{\Gamma'}$ in the natural way.

Let $g : B \to B$ be a continuous map and consider a lift $f : \bar{B} \to \bar{B}$. Note that for every $\gamma \in \Gamma$, the map $f \circ \gamma$ is also a lift of g and therefore there exists a unique $f^{\#}(\gamma) \in \Gamma$ such that $f \circ \gamma = f^{\#}(\gamma) f$. The map $f^{\#} : \Gamma \to \Gamma$ is a group morphism which makes the following diagram commute

$$\begin{tikzcd}
\Gamma \arrow[r, "f^{\#}"] \arrow[d, "\varphi_{b,b_0}"] & \Gamma \arrow[d, "\varphi_{g(b),f(b_0)}"] \\
\pi_1(B,b) \arrow[r, "g_*"] & \pi_1(B,g(b))
\end{tikzcd}$$

for $b \in B$ and $b_0 \in \bar{B}$ such that $p(b_0) = b$. Therefore, under the identification between Γ and the fundamental group, the group morphism $f^{\#}$ corresponds to the induced map g_* on the fundamental group. The advantage of working in this way is that the map $f^{\#}$ does not depend on the choice of a basepoint. The map $f^{\#}$ does depend on the choice of lift of g, but in the cases we consider in this thesis, there will be a canonical choice for the lift. If f is a diffeomorphism, then $g_*(\gamma) = f\gamma f^{-1}$.

Let f be a lift of the map g and consider $\tilde{p} : E \to B$ be another covering map of B. Then there exists a lift $\tilde{f} : E \to E$ of the map $g : B \to B$ if and only if $g_*(\Gamma') \leq \Gamma'$ with $\Gamma' \leq \Gamma$ the subgroup corresponding to the covering map $\tilde{p}$.

2.3 Infra-nilmanifolds

Using the introduction of the previous section, we are ready to introduce infra-nilmanifolds as a generalization of flat closed manifolds. We start by recalling the basic theory of flat manifolds. For more details, we refer to [27].

Flat manifolds

Consider the Euclidean space $\mathbb{R}^n$ which has as isometry group $\text{Iso}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n)$. The isometry group $\text{Iso}(\mathbb{R}^n)$ forms a subgroup of the affine transformations $\text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$ which acts on $\mathbb{R}^n$ in a natural way. The action of an element $F = (b, A) \in \text{Iso}(\mathbb{R}^n)$ on $x \in \mathbb{R}^n$ is given by

$$F(x) = b + A(x).$$

Every subgroup of $\text{Aff}(\mathbb{R}^n)$ also acts on $\mathbb{R}^n$ by restricting the action. We consider $\mathbb{R}^n$ as the subgroup of pure translations of $\text{Iso}(\mathbb{R}^n)$.

A subgroup $\Gamma \leq \text{Iso}(\mathbb{R}^n)$ acts properly discontinuously on $\mathbb{R}^n$ if and only if Γ is a discrete subgroup of $\text{Iso}(\mathbb{R}^n)$. A crystallographic group Γ is a discrete subgroup $\Gamma \leq \text{Iso}(\mathbb{R}^n)$ such that the quotient space $\Gamma \backslash \mathbb{R}^n$ is compact.

A Bieberbach group is a torsion-free crystallographic group $\Gamma \leq \text{Iso}(\mathbb{R}^n)$. In this case the action of Γ on $\mathbb{R}^n$ is free and thus $\Gamma \backslash \mathbb{R}^n$ is a closed manifold. This manifold has Γ as fundamental group and since $\mathbb{R}^n$ is contractible, $\Gamma \backslash \mathbb{R}^n$ is a $K(\Gamma, 1)$ space.

Since Γ acts by isometries, the metric on $\mathbb{R}^n$ induces a metric on the manifold $\Gamma \backslash \mathbb{R}^n$ such that the projection map $\mathbb{R}^n \to \Gamma \backslash \mathbb{R}^n$ is a local isometry. This implies that the manifold $\Gamma \backslash \mathbb{R}^n$ is a flat manifold since $\mathbb{R}^n$ is flat for the euclidean metric. So every Bieberbach group Γ gives rise to a closed flat manifold $\Gamma \backslash \mathbb{R}^n$.

Vice versa, if M is a closed flat manifold then its universal cover $\bar{M}$ is isometric to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. So the fundamental group acts properly discontinuously and freely on the manifold $\mathbb{R}^n$ by isometries and is thus isomorphic to a Bieberbach group Γ. So every closed flat manifold is given by the quotient space $\Gamma \backslash \mathbb{R}^n$ for some Bieberbach group Γ.

Example 2.9. (i) The easiest examples of Bieberbach groups are given by the discrete subgroup $\mathbb{Z}^n \leq \mathrm{Iso}(\mathbb{R}^n)$ of pure translations. In this case, the quotient space is equal to the n-torus $\mathbb{T}^n$.

(ii) The Klein bottle $\mathbb{K}^2$ is a closed flat manifold and its fundamental group is generated by the isometries $a, b \in \mathrm{Iso}(\mathbb{R}^2)$ given by

$$a = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \; b = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

The classical work of Bieberbach, see [8] and [9], determined the algebraic structure of the crystallographic groups Γ. His work is summarized in three theorems which we describe below.

Theorem 2.10 (First Bieberbach Theorem)**.** *Let $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n)$ be a crystallographic group, then the subgroup of pure translations $N = \Gamma \cap \mathbb{R}^n$ is a lattice in $\mathbb{R}^n$ and $[\Gamma : N] < \infty$.*

The finite group $F = {}^{\Gamma}\!/_{N}$ is naturally isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$ and this group is called the holonomy group of the crystallographic group Γ.

This theorem implies in particular that every flat manifold is finitely covered by the torus $N \backslash \mathbb{R}^n$. The group N is a normal subgroup of Γ, so in particular every crystallographic group fits in a short exact sequence

$$1 \longrightarrow N \simeq \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow F \longrightarrow 1 \tag{2.1}$$

where F is the holonomy group. Since the group N is an abelian normal subgroup, the group F has a natural representation $\rho : F \to \mathrm{GL}(n, \mathbb{Z})$ induced by conjugation. The representation ρ is called the holonomy representation and has a geometric meaning, see [105].

Vice versa, every group Γ fitting in a short exact sequence (2.1) is isomorphic to a crystallographic group. So the crystallographic groups are exactly the groups which are virtually $\mathbb{Z}^n$. So the first Bieberbach theorem describes which groups are crystallographic groups.

The second Bieberbach theorem describes the form of the isomorphisms between crystallographic groups.

Theorem 2.11 (Second Bieberbach Theorem)**.** *Let Γ, $\Gamma' \leq \mathrm{Iso}(\mathbb{R}^n)$ be two crystallographic groups and $\varphi : \Gamma \to \Gamma'$ an isomorphism. Then there exists an $\alpha \in \mathrm{Aff}(\mathbb{R}^n)$ such that $\varphi(\gamma) = \alpha\gamma\alpha^{-1}$ for every $\gamma \in \Gamma$.*

So two flat manifolds with isomorphic fundamental group are affinely diffeomorphic, meaning that there is a diffeomorphism between them induced by an affine transformation.

The third Bieberbach theorem states that there are only finitely many flat manifolds in every dimension.

Theorem 2.12 (Third Bieberbach Theorem)**.** *For every $n \in \mathbb{N}$, there are up to affine conjugation only a finite number of crystallographic groups in* $\operatorname{Iso}(\mathbb{R}^n)$*.*

There is also an algorithm to compute the crystallographic groups. In dimension 2, there are exactly 17 crystallographic groups of which exactly 2 are torsion-free. In Example 2.9 we thus gave all examples of flat 2-dimensional manifolds.

Infra-nilmanifolds

The class of infra-nilmanifolds forms a generalization of the flat closed manifolds.

A first way to see this is due to M. Gromov, see [53]. In that paper, the author introduces manifolds which are almost flat in the sense that they admit a family of Riemannian metrics such that the volume of the manifold is bounded but the sectional curvature converges to 0. These manifolds form a geometric generalization of the flat manifolds.

A second way is to generalize the algebraic construction of closed flat manifold as quotient spaces $\Gamma\backslash\mathbb{R}^n$ with Γ a Bieberbach group. The construction above starts from the Euclidean space $\mathbb{R}^n$, where $\mathbb{R}^n$ is the unique simply connected and connected abelian Lie group of dimension n. So by replacing $\mathbb{R}^n$ by more general Lie groups, we find a more general class of manifolds. Since nilpotent groups are groups close to being abelian, this is the logical choice to start from.

Let G be a connected and simply connected nilpotent Lie group. The first step is to generalize the group $\operatorname{Aff}(\mathbb{R}^n)$ of affine transformations of $\mathbb{R}^n$. Let $\operatorname{Aut}(G)$ be the group of continuous automorphisms of G and define the affine group $\operatorname{Aff}(G) = G \rtimes \operatorname{Aut}(G)$. The group $\operatorname{Aff}(G) \curvearrowright G$ in the following natural way:

$$\forall \alpha = (g, \delta) \in \operatorname{Aff}(G),\ \forall h \in G :\ {}^{\alpha}h = g\delta(h).$$

The isometry group $\operatorname{Iso}(\mathbb{R}^n)$ is equal to the semi-direct product $\mathbb{R}^n \rtimes O(n)$. Note that $O(n)$ is a maximal compact subgroup of $\operatorname{GL}(n, \mathbb{R})$. Let C be a maximal compact subgroup of $\operatorname{Aut}(G)$ and consider the subgroup $G \rtimes C$ of $\operatorname{Aff}(G)$. Similarly as in the Euclidean case, we can define a Riemannian metric on G such that $G \rtimes C$ acts by isometries on G.

Again, a subgroup $\Gamma \leq G \rtimes C$ is discrete if and only if Γ acts properly discontinuously on G. An almost-crystallographic group is then a discrete subgroup $\Gamma \leq G \rtimes C$ such that $\Gamma \backslash G$ is compact. If Γ is moreover torsion-free, then the action is free and the quotient space is a closed manifold with fundamental group Γ. In this case, we call Γ an almost-Bieberbach group and $\Gamma \backslash G$ is called an infra-nilmanifold.

Denote by $p : \mathrm{Aff}(G) \to \mathrm{Aut}(G)$ the projection on the second component. The group $F = p(\Gamma)$ is called the holonomy group of Γ. The easiest examples of infra-nilmanifolds which are not flat manifolds are given the ones with trivial holonomy group F. In this case, Γ is just a discrete subgroup of G such that $\Gamma \backslash G$ is compact and we call Γ a uniform lattice in this case. The manifolds we get in this way are called nilmanifolds and are the equivalent of tori in the flat case.

Example 2.13. (i) Every flat manifold is an infra-nilmanifold by taking $G = \mathbb{R}^n$.

(ii) Let R be any commutative ring with unity and consider the Heisenberg group

$$H_3(R) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in R \right\}$$

over the ring R. Take the lattice $H_3(\mathbb{Z})$ in the Lie group $G = H_3(\mathbb{R})$, then the quotient space $H_3(\mathbb{Z}) \backslash H_3(\mathbb{R})$ is a nilmanifold. Similarly, one can consider the groups

$$N_k = \left\{ \begin{pmatrix} 1 & x & \frac{z}{k} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{Z} \right\}$$

for $k > 0$ which form lattices in $H_3(\mathbb{R})$.

(iii) Consider the automorphism $\psi : H_3(\mathbb{R}) \to H_3(\mathbb{R})$ given by

$$\psi\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -x & z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

The automorphism ψ has order 2 and let $F \leq \mathrm{Aut}(H_3(\mathbb{R}))$ be the subgroup generated by ψ. Consider the subgroup Γ of $H_3(\mathbb{R}) \rtimes F$ generated by $H_3(\mathbb{Z})$ and the element

$$\left(\begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \psi \right) \in \mathrm{Aff}(H_3(\mathbb{R})).$$

It is an exercise to check that Γ is torsion-free and thus forms an almost-Bieberbach group.

The algebraic structure of the crystallographic groups was described by the Bieberbach theorems. In the next section we show that these theorems have a nice generalization to the almost-crystallographic groups.

2.4 Algebraic structure of almost-crystallographic groups

As discussed in the previous section, the Bieberbach theorems describe the algebraic structure of crystallographic groups. In this section we discuss the generalization of these results to almost-crystallographic groups. We start by describing some properties of lattices of nilpotent Lie groups G.

Nilpotent groups

There are a few properties of nilpotent groups which are useful in this thesis. For the first property, we first introduce subnormal subgroups.

Definition 2.14. Let G be a group, then we call a subgroup $H \leq G$ **subnormal** if there exists a sequence of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_m = G$$

with H_j a normal subgroup of H_{j+1}. The smallest possible m such that such a sequence exists is called the subnormal depth of H.

In abelian groups, every subgroup is a normal subgroup and thus every subgroup is also subnormal. By applying this fact on nilpotent groups and using induction, we also find that every subgroup of a nilpotent group is subnormal.

Theorem 2.15. *Every subgroup of a nilpotent group is subnormal. Moreover, the subnormal depth is uniformly bounded by the nilpotency class.*

Another property of torsion-free abelian groups is that if $x^n = y^n$ for some $n > 0$, then also $x = y$. Again, by using induction and this result for abelian groups, we get the following.

Theorem 2.16. *Let G be a torsion-free nilpotent group and assume that $g^n = h^n$ for some $n > 0$, then $x = y$.*

For the Lie group $\mathbb{R}^n$, every uniform lattice is isomorphic to the discrete group $\mathbb{Z}^n$ and therefore every flat manifold with trivial holonomy group is a torus $\mathbb{T}^n$. Note that $\mathbb{Z}^n$ is the free abelian group on n generators.

Every uniform lattice of G is a finitely generated torsion-free nilpotent group. Groups satisfying these three properties are called $\mathcal{F}$-groups. Conversely, every $\mathcal{F}$-group is isomorphic to a uniform lattice in some connected and simply connected nilpotent Lie group G. In particular, all properties described above are valid for $\mathcal{F}$-groups.

Generalized Bieberbach theorems

The first Bieberbach theorem states that the subgroup of pure translations is a lattice in the covering Lie group and that the holonomy group is finite. The statement in the nilpotent case is identical.

Theorem 2.17 (Generalized first Bieberbach theorem)**.** *Let $\Gamma \leq \mathrm{Aff}(G)$ be an almost-crystallographic group and consider $N = \Gamma \cap G$ the subgroup of pure translations. Then N is a uniform lattice in the Lie group G and the holonomy group $F = {}^{\Gamma}\!\diagup_{N}$ is finite.*

So in particular, every infra-nilmanifold $\Gamma \backslash G$ is finitely covered by the nilmanifold $N \backslash G$, just as every flat manifold is finitely covered by a torus. In fact, N is equal to the Fitting subgroup of Γ, i.e. the maximal nilpotent normal subgroup of Γ. The holonomy group is isomorphic to a finite subgroup of $\mathrm{Aut}(G)$.

Every almost-Bieberbach group therefore fits in an exact sequence

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow F \longrightarrow 1 \tag{2.2}$$

where N is an $\mathcal{F}$-group. Conversely, every group Γ fitting in an exact sequence (2.2) is an almost-crystallographic group.

The generalization of the second Bieberbach theorem also describes the isomorphisms between almost-crystallographic groups.

Theorem 2.18 (Generalized second Bieberbach theorem)**.** *Let $\varphi : \Gamma_1 \to \Gamma_2$ be an isomorphism between two almost-crystallographic groups $\Gamma_i \leq \mathrm{Aff}(G)$ modeled on the same Lie group G. Then there exists an affine transformation $\alpha \in \mathrm{Aff}(G)$ such that $\varphi(\gamma) = \alpha\gamma\alpha^{-1}$ for every $\gamma \in \Gamma_1$.*

Note that if $\delta \in \mathrm{Aut}(G)$, then $\delta g \delta^{-1} = \delta(g)$ for all $g \in G$.

Finally, the third generalized Bieberbach theorem tries to describe the number of different almost-crystallographic groups. First consider the following example.

Example 2.19. Consider the groups N_k of Example 2.13. For each of these groups, the commutator subgroup $[N_k, N_k]$ is generated by the element

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and thus the torsion subgroup of ${}^{N_k}\!/\!_{[N_k, N_k]}$ is isomorphic to the cyclic group $\mathbb{Z}_k$. This shows that these groups are pairwise non-isomorphic.

Theorem 2.20 (Generalized third Bieberbach theorem)**.** *Up to isomorphism there are only a finite number of almost-crystallographic groups which have isomorphic Fitting subgroup.*

Self-maps on infra-nilmanifolds

To end this section we describe some classes of self-maps on infra-nilmanifolds.

Let $\Gamma \leq \mathrm{Aff}(G)$ be an almost-Bieberbach group modeled on the Lie group G. Every injective group morphism $\varphi : \Gamma \to \Gamma$ of Γ is given by conjugation by an affine transformation $\alpha \in \mathrm{Aff}(G)$, so $\varphi(\gamma) = \alpha\gamma\alpha^{-1}$.

Vice versa, assume that $\alpha \in \mathrm{Aff}(G)$ satisfies $\alpha\Gamma\alpha^{-1} \leq \Gamma$. The affine transformation α induces a differentiable map

$$\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G : g \mapsto {}^{\alpha}g,$$

which is indeed well-defined since

$${}^{\alpha}\left({}^{\gamma}g\right) = {}^{\alpha\gamma}g = {}^{\alpha\gamma\alpha^{-1}}\left({}^{\alpha}g\right) \in \Gamma^{\alpha}g.$$

The map $\bar{\alpha}$ is called an affine infra-nilmanifold endomorphism. If $\Gamma\backslash G$ is a nilmanifold, then we say that $\bar{\alpha}$ is an affine nilmanifold endomorphism. If $\alpha \in \mathrm{Aut}(G)$, then we call $\bar{\alpha}$ an infra-nilmanifold endomorphism. In particular, a nilmanifold endomorphism is a map $\bar{\delta}$ induced by an automorphism $\delta \in \mathrm{Aut}(G)$ on a nilmanifold $\Gamma\backslash G$.

If $\alpha\Gamma\alpha^{-1} = \Gamma$, then also the affine transformation α^{-1} induces a differentiable map $\Gamma\backslash G \to \Gamma\backslash G$ which is the inverse of $\bar{\alpha}$. This implies that $\bar{\alpha}$ is a diffeomorphism and therefore $\bar{\alpha}$ is called an affine infra-nilmanifold automorphism in that case.

From Section 2.1 it follows that automorphisms of the Lie group G correspond to automorphisms of the Lie algebra $\mathfrak{g}$ corresponding to G. The eigenvalues of an automorphism $\mathrm{Aut}(G)$ are defined as the eigenvalues of the corresponding

Lie algebra automorphism. The eigenvalues of an affine infra-nilmanifold endomorphism $\bar{\alpha}$ is then defined as the eigenvalues of the linear part of α.

The affine transformation α is a lift of the affine infra-nilmanifold endomorphism $\bar{\alpha}$. Since α is a diffeomorphism, this implies that $\bar{\alpha}_* : \Gamma \to \Gamma$ is given by conjugation by α. By Theorem 2.8 and Theorem 2.18 this implies that every map $f : \Gamma\backslash G \to \Gamma\backslash G$ on an infra-nilmanifold with $f_* : \Gamma \to \Gamma$ injective is homotopic to an affine infra-nilmanifold endomorphism. There is a similar result for general self-maps on infra-nilmanifolds by considering a bigger class of self-maps.

In Chapter 7 we also study the periodic points of a more general of self-maps. Let $\mathrm{Endo}(G)$ be the semi-group of group morphisms $G \to G$ and consider the semi-group $\mathrm{aff}(G) = G \rtimes \mathrm{Endo}(G)$ of affine maps on G which are not necessarily invertible. The semi-group $\mathrm{aff}(G)$ also acts in a natural way on G by extending the natural action of $\mathrm{Aff}(G)$ on G, so if $\alpha = (g, \delta)$, then

$$^{\alpha}h = g\delta(h)$$

for every $h \in G$.

Let $\alpha \in \mathrm{aff}(G)$ be an affine map such that $\alpha\Gamma \subseteq \Gamma\alpha$, or equivalently such that for every $\gamma \in \Gamma$, there exists some $\gamma' \in \Gamma$ with $\alpha\gamma = \gamma'\alpha$. The map α will induce a map

$$\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$$

given by $\bar{\alpha}(\Gamma g) = \Gamma {}^{\alpha}g$ and the map $\bar{\alpha}$ is called a generalized affine infra-nilmanifold endomorphism.

The generalization of Theorem 2.18 to arbitrary group morphisms of almost-crystallographic groups is the following.

Theorem 2.21 ([78]). *Let $\varphi : \Gamma \to \Gamma$ be a group morphism of an almost-crystallographic group Γ. Then there exists $\alpha \in \mathrm{aff}(G)$ such that*

$$\varphi(\gamma)\alpha = \alpha\gamma$$

for all $\gamma \in \Gamma$.

Similarly as above, it holds that every map $f : \Gamma\backslash G \to \Gamma\backslash G$ is homotopic to a map induced by an element $\alpha \in \mathrm{aff}(G)$.

2.5 Rational holonomy representation

In this section we introduce the rational holonomy representation of an almost-Bieberbach group $\Gamma \leq \mathrm{Aff}(G)$. This representation generalizes the holonomy representation $F \to \mathrm{GL}(n, \mathbb{Z})$ of flat manifolds to infra-nilmanifolds.

A first step in defining this representation is to define the radicable hull $N^{\mathbb{Q}}$ of the Fitting subgroup $N \lhd \Gamma$. The rational holonomy group is then a representation of the finite holonomy group $F = {}^{\Gamma}\!/\!_{N}$ of the form

$$\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$$

such that Γ is naturally isomorphic to a subgroup of $N^{\mathbb{Q}} \rtimes_{\rho} F$.

Radicable hull

Let N be the Fitting subgroup of the almost-Bieberbach group of Γ, so

$$N = \Gamma \cap G \leq G$$

is equal the subgroup of pure translations. The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism and we denote its inverse by $\log : G \to \mathfrak{g}$. The set $\log(N) \subseteq \mathfrak{g}$ forms a discrete subset of the Lie algebra $\mathfrak{g}$ which is in general not closed under addition in the Lie algebra $\mathfrak{g}$.

The rational span of $\log(N)$, denoted as $\mathbb{Q}\log(N) = \mathfrak{n}^{\mathbb{Q}}$, is a rational Lie subalgebra of $\mathfrak{g}$, see [93, Theorem 6.2]. From the Baker-Campbell-Hausdorff formula it follows that $N^{\mathbb{Q}} = \exp(\mathfrak{n}^{\mathbb{Q}})$ is a subgroup of G. The Lie group G is nilpotent and torsion-free and therefore also the group $N^{\mathbb{Q}}$ is a torsion-free and nilpotent group as a subgroup of G.

Since $\exp(v)^k = \exp(kv)$, we get that every $n \in N^{\mathbb{Q}}$ has some power n^k such that $n^k \in N$. For every $n = \exp(v) \in N^{\mathbb{Q}}$ and $k \in \mathbb{N}$, there exists an $m \in N^{\mathbb{Q}}$ such that $m^k = n$, namely the element $\exp(\frac{v}{k})$. A group satisfying this property is called radicable. Note that since $N^{\mathbb{Q}}$ is nilpotent, Theorem 2.16 implies that the element m is always unique in this definition. Any other group satisfying these properties is isomorphic to $N^{\mathbb{Q}}$ by an isomorphism fixing N pointwise.

Definition 2.22. Let N be a $\mathcal{F}$-group, then the **radicable hull** of N, denoted as $N^{\mathbb{Q}}$, is the unique nilpotent, torsion-free and radicable group which contains N as subgroup such that every $n \in N^{\mathbb{Q}}$ has some power $n^k \in N$.

The existence of a radicable hull thus follows from the construction above.

The radicable hull $N^{\mathbb{Q}}$ does not uniquely determine the $\mathcal{F}$-group N, for example finite index subgroups have the same radicable hull.

Proposition 2.23. *Let N_1 be an $\mathcal{F}$-group and $N_1 \leq N_2$ a subgroup of finite index, then $N_1^{\mathbb{Q}}$ is also the radicable hull of N_2.*

Proof. The group $N_1^{\mathbb{Q}}$ is torsion-free, nilpotent and radicable, so it suffices to check that every element has some power lying in N_2. This is immediate, since N_2 is a subgroup of finite index. □

So $\mathcal{F}$-groups which are abstractly commensurable have the same radicable hull. The converse of this statement is true as well.

Theorem 2.24. *Let N_1 and N_2 $\mathcal{F}$-groups, then N_1 and N_2 are abstractly commensurable if and only if $N_1^{\mathbb{Q}}$ and $N_2^{\mathbb{Q}}$ are isomorphic.*

Every radicable torsion-free nilpotent group $N^{\mathbb{Q}}$ corresponds to some rational nilpotent $\mathfrak{n}^{\mathbb{Q}}$ Lie algebra by the Baker-Campbell-Hausdorff formula. The groups $\operatorname{Aut}(N^{\mathbb{Q}})$ and $\operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ are naturally isomorphic via the exponential map and sometimes we will identify these two groups without further mentioning the exponential map.

A finitely generated subgroup N of $N^{\mathbb{Q}}$ such that $N^{\mathbb{Q}}$ is the radicable hull of N is called a full subgroup. Every radicable torsion-free nilpotent group $N^{\mathbb{Q}}$ of finite dimension has a full subgroup, for example the group generated by $\exp(v_1)$, $\ldots$, $\exp(v_n)$ where $v_1, \ldots, v_n$ is a basis for $\mathfrak{n}^{\mathbb{Q}}$. In particular, every rational nilpotent Lie algebra corresponds to some $\mathcal{F}$-group N.

Every injective group morphism $\varphi : N \to N$ has a unique extension to an automorphism of $N^{\mathbb{Q}}$, which we denote as $\varphi^{\mathbb{Q}}$. The automorphism $\varphi^{\mathbb{Q}}$ corresponds to a Lie algebra automorphism of $\mathfrak{n}^{\mathbb{Q}}$. The eigenvalues of φ are then the eigenvalues of $\varphi^{\mathbb{Q}}$ as Lie algebra automorphism.

Rational holonomy representation

From the exact sequence (2.2), it follows that the group Γ contains a normal subgroup of finite index which is an $\mathcal{F}$-group, but this exact sequence does not split in general. We can embed the exact sequence (2.2) in a split exact sequence.

Starting from the exact sequence (2.2) and using the fact that every automorphism of N has a unique extension to $N^{\mathbb{Q}}$, we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, hook] & \Gamma \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N^{\mathbb{Q}} \arrow[r] & \Gamma^{\mathbb{Q}} \arrow[r] & F \arrow[r] & 1.
\end{tikzcd}$$

The group $\Gamma^{\mathbb{Q}}$ is the pushout of the groups $N^{\mathbb{Q}}$ and Γ with respect to the subgroup N. The lower sequence is also exact and it follows from [27] that this sequence splits. By fixing a splitting morphism $s : F \to \Gamma^{\mathbb{Q}}$, we get an injective morphism $F \to \mathrm{Aut}(N^{\mathbb{Q}})$. This morphism is called the rational holonomy representation. The group Γ is by definition isomorphic to a subgroup of $\Gamma^{\mathbb{Q}} \simeq N^{\mathbb{Q}} \rtimes_\rho F$. The representation $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$ is always injective and sometimes we will identify the group F with its image under this representation.

The rational holonomy representation encodes some, but not all information about the geometry and topology of the infra-nilmanifold. We give some examples in the case of flat manifolds.

Example 2.25. Let $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ be a crystallographic group with rational holonomy representation $\rho : F \to \mathrm{GL}(n, \mathbb{Q})$.

(i) There is a criterion to check whether the group $\mathrm{Out}(\Gamma)$ is finite which depends only on the representation ρ, see [102].

(ii) If Γ is a Bieberbach group, then the representation ρ also determines if the manifold $\Gamma\backslash\mathbb{R}^n$ is Kähler flat or not by [26].

(iii) In [64], there are examples of flat manifolds $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ such that both have the same rational holonomy representation but $\Gamma_1\backslash\mathbb{R}^n$ admits a spin structure and $\Gamma_2\backslash\mathbb{R}^n$ does not admit one.

The existence of an Anosov diffeomorphism depends only on this representation, as we will discuss in Theorem 3.36. The main result of Chapter 6 states that the same is true for expanding maps and non-trivial self-covers of infra-nilmanifolds.

By using the rational holonomy representation, we have the following version of the Generalized Second Bieberbach Theorem:

Theorem 2.26. *Let Γ and Γ' be two almost-Bieberbach groups with abstractly commensurable Fitting subgroups N and N' and identify their radicable hulls $N^{\mathbb{Q}} = (N')^{\mathbb{Q}}$. If $\varphi : \Gamma \to \Gamma'$ is an isomorphism then there exists an affine transformation $\alpha \in N^{\mathbb{Q}} \rtimes \mathrm{Aut}(N^{\mathbb{Q}})$ such that $\varphi(\gamma) = \alpha\gamma\alpha^{-1}$.*

The proof of this version of the Second Bieberbach Theorem is identical to the proof given in [27, Section 2.2] by replacing the Lie group G by the radicable hull $N^{\mathbb{Q}}$. Note that the theorem also works for injective group morphisms $\Gamma \to \Gamma$, since the image is a subgroup of finite index in that case.

Note that the construction of $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$ is similar to the proof of showing that torsion-free virtually nilpotent groups are almost-Bieberbach groups. The same construction can be repeated for every field E and even for some rings containing $\mathbb{Z}$.

Lattice hull

To end this section, we introduce lattice subgroups of $N^{\mathbb{Q}}$. First, consider the following classical example.

Example 2.27. Consider the real nilpotent Lie algebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\} \subseteq \mathbb{R}^{3\times 3}$$

which corresponds to the Lie group $H_3(\mathbb{R})$. The exponential mapping of $\mathfrak{h}$ is given by

$$\exp\left(\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

This implies that $\log(H_3(\mathbb{Z}))$ is not closed under addition in the Lie algebra $\mathfrak{h}$, although it is closed under the multiplication defined by the Baker-Campbell-Hausdorff formula. On the other hand, $H_3(\mathbb{Z})$ is a subgroup of finite index in the group N_2 of Example 2.13 and $\log(N_2)$ is closed under addition.

Consider a $\mathcal{F}$-group N which is a lattice in some nilpotent Lie group G with diffeomorphism $\log : G \to \mathfrak{g}$. Example 2.27 shows the set $\log(N)$ is not a Lie subring or even an additive subgroup of $\mathfrak{g}$. Therefore the following definition is important.

Definition 2.28. An $\mathcal{F}$-group N is called a **lattice group** if $\log(N)$ is an additive subgroup of $\mathfrak{g}$.

This means that for a lattice group N we can always find a basis for $\mathfrak{n}^{\mathbb{Q}}$ such that $\log(N)$ is equal to the $\mathbb{Z}$-span of this basis.

The existence of lattice subgroups is non-trivial and uses the structure of the Baker-Campbell-Hausdorff formula. For example, the following is shown in [93, Proposition 6.1].

Proposition 2.29. *Let N be an $\mathcal{F}$-group with radicable hull $N^{\mathbb{Q}}$. Then there exists a lattice subgroup $N' \leq N^{\mathbb{Q}}$ such that N is a subgroup of finite index in N'.*

Since the intersection of lattice groups is again a lattice group, there exists a smallest lattice group containing N. This smallest lattice group containing N is called the lattice hull of N and is denoted as N^{lat}. Let $\varphi : N \to N$ be an injective group morphism and consider the extension $\varphi^{\mathbb{Q}} \in \mathrm{Aut}(N^{\mathbb{Q}})$. Since $\varphi^{\mathbb{Q}}$ maps lattice groups to lattice groups, the following proposition is immediate (see also [7, Lemma 4.1.]).

Proposition 2.30. *Let N be an $\mathcal{F}$-group and $\varphi : N \to N$ an injective group morphism. Then the following are true:*

1. $\varphi^{\mathbb{Q}}(N^{\mathrm{lat}}) = (\varphi(N))^{\mathrm{lat}} \leq N^{\mathrm{lat}}$;
2. $[N : \varphi(N)] = [N^{\mathrm{lat}} : \varphi^{\mathbb{Q}}(N^{\mathrm{lat}})] = |\det \varphi^{\mathbb{Q}}| = |\det \varphi|$.

The first statement shows that every injective group morphism of N also induces an injective group morphism on N^{lat} and we will denote the induced map as φ^{lat}.

Chapter 3

Expanding maps and Anosov diffeomorphisms

This chapter gives an introduction to dynamical systems with an emphasis on expanding maps and Anosov diffeomorphisms. These self-maps of manifolds are intensively studied because of their interesting dynamical properties, in particular their chaotic behavior and structural stability.

First, we introduce some dynamical properties of self-maps on metric spaces which are interesting to study. Next, we define expanding maps and Anosov diffeomorphisms on closed manifolds and give some examples on infra-nilmanifolds. It is conjectured that the class of infra-nilmanifolds is up to homeomorphism the only one admitting expanding maps and Anosov diffeomorphisms. We also give an overview of the known results about the existence of expanding maps and Anosov diffeomorphisms on infra-nilmanifolds. Finally, we introduce the topological counterpart of these maps and show how they can be studied in essentially the same way.

3.1 Dynamical systems

In dynamical systems, one starts from a continuous family of self-maps

$$f_t : X \to X$$

on some metric space (X, d) and studies the global orbit structure of this family. In this thesis we assume the space X to be compact and connected and in many cases X is a closed Riemannian manifold. Our main focus is on discrete dynamical systems, which are given by a single self-map $f : X \to X$ and the continuous family is then given by the iterations $f^n = f \circ \ldots \circ f$ of the map f. The global orbit structure in this case describes how points move in the set X under these iterations f^n.

This section recalls some properties about discrete dynamical systems which are studied in literature, including structural stability and chaotic behavior. Most properties we are interested in are invariant under coordinate transformations and we start by making exact what this means. A more detailed introduction can be found in the standard references [44] or [70].

Topological conjugacy

We are interested in properties of dynamical systems which are invariant under transformations of the space X. These transformations could be homeomorphisms, diffeomorphisms or even an isomorphism of measure spaces, depending on the structure of the space X. Although there are strong connections between ergodic theory and Anosov diffeomorphisms, we don't go into details about measure theory.

In our case, the most useful equivalence relation when studying dynamical systems is topological conjugation.

Definition 3.1. Let $f : X \to X$ and $g : Y \to Y$ be self-maps of the topological spaces X and Y. The maps f and g are **topologically conjugate** if there exists a homeomorphism $h : X \to Y$ which makes the following diagram commutative.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] & Y \arrow[d, "g"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

The homeomorphism h is called a **topological conjugation** between f and g.

Note that if h is a topological conjugation between f and g, then h is also a topological conjugation between the iterations f^n and g^n since

$$h \circ f^n = h \circ f \circ f^{n-1} = g \circ h \circ f^{n-1} = \ldots = g^n \circ h. \tag{3.1}$$

If two maps f and g are topologically conjugate, then these maps have the same dynamical properties from the topological viewpoint. For example, the orbits of

f are mapped homeomorphically to orbits of g by the topological conjugation h.

Since expanding maps and Anosov diffeomorphisms are only defined on differentiable manifolds, it seems unnatural to only assume that h is a homeomorphism in Definition 3.1. Indeed, if X and Y are manifolds, there is also the notion of differentiable conjugation where the homeomorphism h in Definition 3.1 is replaced by a diffeomorphism. This notion is much too rigid though for studying dynamical systems, since in most cases a classification of certain self-maps up to differentiable conjugacy is impossible. Also, every interesting dynamical property is already preserved by topological conjugacy, so there is no need to restrict to this stronger equivalence relation. For example, if h is a differentiable conjugation between f and g, then the linear maps $Df_x : T_xM \to T_{f(x)}M$ and $Dg_{h(x)} : T_{h(x)}N \to T_{g(h(x))}N$ are conjugate by dh_x. This is a strong condition which doesn't give us any information about the dynamical properties of f and g.

Periodic points

The first interesting property for self-maps $f : X \to X$ on a metric space X to study is the set of periodic points. In Chapter 7 for example we investigate the periodic and eventually periodic points for a general class of self-maps on infra-nilmanifolds.

A fixed point $x \in X$ of f is a point satisfying $f(x) = x$. We write $\mathrm{Fix}(f)$ for the set of all fixed points of f. We call $x \in X$ periodic if there exists some iteration f^n of f such that x is a fixed point of f^n. The subset of all periodic points of f is denoted as $\mathrm{Per}(f) \subseteq X$. A point $x \in X$ is called eventually periodic if $f^n(x)$ is periodic for some $n > 0$ and the set of eventually periodic points is denoted as $\mathrm{ePer}(f)$.

We have the natural inclusions

$$\mathrm{Fix}(f) \subseteq \mathrm{Per}(f) \subseteq \mathrm{ePer}(f)$$

of subsets in X. From the definition, it follows that $\mathrm{Per}(f) = \mathrm{Per}(f^n)$ and $\mathrm{ePer}(f) = \mathrm{ePer}(f^n)$ for every $n > 0$. It also holds that

$$\mathrm{Per}(f) = \bigcup_{n \in \mathbb{N}_0} \mathrm{Fix}(f^n).$$

If two maps are topologically conjugate, there is an immediate relation between their sets of (eventually) periodic points.

Proposition 3.2. *Let $f : X \to X$ and $g : Y \to Y$ be self-maps of topological spaces X and Y. Assume that $h : X \to Y$ is a topologically conjugation between f and g. Then the following equalities hold:*

$$h(\mathrm{Fix}(f)) = \mathrm{Fix}(g)$$

$$h(\mathrm{Per}(f)) = \mathrm{Per}(g)$$

$$h(\mathrm{ePer}(f)) = h(\mathrm{ePer}(g)).$$

The proof is immediate from equation (3.1).

Proposition 3.2 shows that it suffices to study the (eventually) periodic points of one self-map in every equivalence class under topological conjugacy. In Chapter 7 this observation is used to determine the (eventually) periodic points of affine infra-nilmanifold endomorphisms.

Non-wandering points form a generalization of periodic points. The non-wandering set $\Omega(f)$ is defined as the set

$$\Omega(f) = \{x \in X \mid \forall U \ni x \text{ open}, \exists n > 0 \text{ with } f^n(U) \cap U \neq \varnothing\}.$$

So non-wandering points are points which are close to being periodic. The periodic points obviously form a subset of the non-wandering set, so

$$\mathrm{Per}(f) \subseteq \Omega(f).$$

The non-wandering set is a closed subset of X which is invariant under the map f. Again, the non-wandering set behaves well under topological conjugation.

Proposition 3.3. *Let $f : X \to X$ and $g : Y \to Y$ be self-maps of topological spaces X and Y and h is a topological conjugation between f and g, then*

$$h(\Omega(f)) = \Omega(g).$$

Interesting dynamical systems have a non-wandering set $\Omega(f)$ such that $\mathrm{Per}(f)$ is dense in $\Omega(f)$ or such that $\Omega(f) = X$. If $\mathrm{Per}(f)$ is dense in $\Omega(f)$, then every point which is close to being periodic is also close to a periodic point.

Chaos

Chaotic behavior for dynamical systems occurs in many practical applications. Although chaotic dynamical systems are deterministic, meaning that the map $f : X \to X$ is completely known, small changes in the initial conditions cause a totally different outcome. These dynamical systems are characterized by three properties which we introduce below.

The first property for chaos is topologically transitivity.

Definition 3.4. Let $f : X \to X$ be a self-map of a topological space X. The map f is called **topologically transitive** if for all non-empty open sets $U, V \subseteq X$, there exists some $n \in \mathbb{N}$ such that

$$f^n(U) \cap V \neq \varnothing.$$

Note that if $f : X \to X$ is topologically transitive, then this implies that $\Omega(f) = X$ by taking $U = V$ in the definition. Intuitively, being topologically transitive means that some points of an arbitrarily small neighbourhood are eventually mapped to every other small neighbourhood by the map f. Loosely said, this means that there are some points that keep on moving through the entire space X.

A stronger property than being topologically transitive is topologically mixing. We say that f is topologically mixing if for all non-empty open sets $U, V \subseteq X$, there exists some $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$ for all $n \geq N$. For the dynamical systems we study in this thesis, topological transitivity implies topologically mixing.

If X is a compact metric and $f : X \to X$ is topologically transitive, then f always has a dense orbit, see [98]. Sometimes topological transitivity is even defined as this property. These definitions are independent for general metric spaces, as follows from the following examples.

Example 3.5. Consider the continuous map $f : [0, 1] \to [0, 1]$ defined as

$$f(x) = 1 - |2x - 1|$$

for all $x \in [0, 1]$. The periodic points of f are dense in $[0, 1]$ and thus the map has infinitely many periodic points. For every open subset $U \subseteq [0, 1]$ it holds that there exists some $n \in \mathbb{N}$ such that $f^n(U) = [0, 1]$.

Now consider the map $g = f|_{\mathrm{Per}(f)}$. The map g only has finite orbits and thus there doesn't exist an orbit which is dense. The map g is topologically transitive though since also for g satisfies $g^n(U) = \mathrm{Per}(f)$ for every open U and some $n > 0$.

Example 3.6. Consider the space

$$X = \{0\} \cup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$$

with the restricted metric from $\mathbb{Q}$ and the continuous map $f : X \to X$ defined by $f(0) = 0$ and

$$f\left(\frac{1}{n}\right) = \frac{1}{n+1}.$$

The point $1 \in X$ has a dense orbit under f, but f is not topologically transitive, for example by taking the opens $U = \left\{\frac{1}{2}\right\}$ and $V = \{1\}$.

The problem in Example 3.6 is the existence of many isolated points. In fact, if X has no isolated points, then there is a relation between the two definitions.

Proposition 3.7. *If X has no isolated points then the existence of a dense orbit implies topological transitivity.*

Proof. Let $x \in X$ be the point with a dense orbit. Take opens U, V, then there exists some $n \in \mathbb{N}$ such that $f^n(x) \in U$. Since X has no isolated points, the set

$$W = V \setminus \{x, f(x), \dots, f^n(x)\}$$

is a non-empty open set of X. Therefore, there exists $m \in \mathbb{N}$ such that $f^m(x) \in W$. By construction of the set W, $m > k$ and thus $f^{m-k}(U) \cap V \neq \varnothing$. $\square$

In particular, since connected topological spaces have no isolated points and we always assume X to be compact and connected, both definitions are equivalent for the spaces we study. More details about the different definitions for topological transitivity can be found in [98].

Definition 3.8. Let $f : X \to X$ be a self-map of a metric space (X, d). We say that f has **sensitive dependence on initial conditions** if there exists some $\epsilon > 0$ such that for every $\delta > 0$ and every $x \in X$, there exists some $y \in X$ such that $d(x, y) < \delta$ but

$$d(f^n(x), f^n(y)) > \epsilon$$

for some $n > 0$.

Intuitively a map f has sensitive dependence on initial conditions if points close to each other can behave completely different under iterations of f. The ϵ in the definition is a parameter controlling the difference in outcome.

Chaotic maps now combine the previous properties with density of the periodic points.

Definition 3.9. Let $f : X \to X$ be a self-map on a metric space (X, d). We call the map f **chaotic** or say that f has **chaotic behavior** if

(1) f is topologically transitive;

(2) f has sensitive dependence on initial conditions;

(3) $\mathrm{Per}(f)$ forms a dense subset of the space X.

Note that this last property implies that $\mathrm{Per}(f)$ is dense in $\Omega(f)$ and $\Omega(f) = X$.

So chaotic dynamical systems combine three totally different properties. Being topologically transitive in the definition can be considered as being indecomposable, since there are no two disjoint proper open subsets which are invariant under the map f. The sensitive dependence on initial conditions states that the dynamical system is unpredictable. Finally the density of periodic points adds some regularity in the dynamical system.

Chaotic behavior is invariant under topological conjugacy.

Theorem 3.10. *Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of metric spaces X and Y which are topologically conjugate. Then f is chaotic if and only if g is chaotic.*

We give a few examples of dynamical systems which satisfy some of these properties. The main examples of chaotic dynamical systems are expanding maps and Anosov diffeomorphism on infra-nilmanifolds.

Example 3.11. Consider the unit circle S^1 of the complex plane $\mathbb{C}$, so

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}$$

which is a Riemannian manifold by restricting the Riemannianm metric of $\mathbb{C}$. Take $\theta \in \mathbb{R}$ and take the homeomorphism $R_\theta : S^1 \to S^1$ defined as

$$R_\theta(z) = e^{2\pi i\theta} z.$$

Note that the subgroup of the multiplicative group S^1 generated by $e^{2\pi i\theta}$ is finite or dense in S^1, depending on the fact if $\theta \in \mathbb{Q}$ or not. If $\theta \in \mathbb{Q}$, then R_θ has finite order and thus is not topologically transitive.

If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then R_θ is topologically transitive since for every $\epsilon > 0$ and $z_1, z_2 \in S^1$, there exists some $k > 0$ such that

$$d(R_\theta^k(z_1), z_2) < \epsilon.$$

The map R_θ is an isometry, so

$$d(R_\theta(z_1), R_\theta(z_2)) = d(z_1, z_2)$$

and thus R_θ doesn't have sensitive dependence on initial conditions. The periodic points of R_θ are equal to S^1 or $\varnothing$, depending whether $\theta \in \mathbb{Q}$ or not. So for $\theta \notin \mathbb{Q}$ we have a dynamical system which is topologically transitive but satisfies no other property of Definition 3.9.

Example 3.12. Let $X = [0, 1]$ the unit interval and let $f : X \to X$ be the map given by

$$f(x) = 4x(1 - x)$$

for all $x \in X$. The map f is chaotic, see [44].

Structural stability

The intuitive idea behind structural stability of a dynamical system $f : X \to X$ is that small perturbations of f have the same dynamical properties as f itself, see Example 1.1 and 1.2. To make this statement exact, we first explain what small perturbations are by introducing the C^1 topology on self-maps of manifold M.

Consider the set $C^0(M)$ of continuous functions $f : M \to M$ and $C^\infty(M)$ of differentiable functions $f : M \to M$. If M is a Riemannian manifold with induced metric d, the C^0 topology on $C^0(M)$ is defined by the metric

$$d_0(f, g) = \sup_{m \in M} \{d(f(m), g(m)\}.$$

Every element of $C^\infty(M)$ induces a continuous map $Df : TM \to TM$ and thus we get an injective map $C^\infty(M) \hookrightarrow C^0(TM)$. The C^1 topology is then defined by restricting the C^0 topology of $C^0(TM)$ to the image of this injective map.

Example 3.13. Consider the maps $f, g : (0, \infty) \to \mathbb{R}$ given by $f(x) = 0$ and $g(x) = \epsilon \sin\left(\frac{1}{x}\right)$ for all $x \in (0, \infty)$. The maps f and g satisfy $d_0(f, g) = \epsilon$ but f and g are far apart in the C^1 topology.

Structural stability is expressed in terms of the C^1 topology.

Definition 3.14. A map $f : M \to M$ of a closed manifold is called **structural stable** if there exists an open subset $U \subseteq C^\infty(M)$ for the C^1 topology such that every element of U is topologically conjugate to the map f.

Unfortunately, structural stability is not invariant under topological conjugations. This is due to the fact that a topological conjugation between two self-maps gives us no information about the C^1 topology.

Example 3.15. For every hyperbolic toral automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ and $\epsilon > 0$, there exists a diffeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ with $d_0(f, g) < \epsilon$ but g not an Anosov diffeomorphism, see [3]. The map g is topologically conjugate to f, but in every neighbourhood of g in the C^1 topology there are many diffeomorphisms which are not topologically conjugate to f.

3.2 Expanding maps

The first type of dynamical systems we consider are expanding maps. We give the definition, some examples and an overview of the known results about their dynamical properties and the manifolds supporting such an expanding map.

What is an expanding map?

The definition of an expanding map generalizes Example 1.1.

Definition 3.16. Let M be a closed manifold with Riemannian metric $\| \cdot \|$. A differentiable map $f : M \to M$ is called **expanding** if there exist constants $c > 0,\ \lambda > 1$ such that for every tangent vector $v \in TM$ we have that

$$\|Df^n(v)\| \geq c\lambda^n \|v\|$$

for every $n \in \mathbb{N}$.

If $\| \cdot \|_1$ and $\| \cdot \|_2$ are two Riemannian metrics on a closed manifold M, there exist constants $0 < c_1 < c_2$ such that

$$c_1 \|v\|_1 \leq \|v\|_2 \leq c_2 \|v\|_1 \quad \forall v \in TM.$$

This implies that the definition of an expanding map does not depend on the choice of Riemannian metric on the manifold M by varying the value of the constant c in the definition. Thus expanding maps can be considered as a topological type of self-map, an idea we pursue in Section 3.5.

In fact, we can always find a Riemannian metric on M such that we can choose $c = 1$ in Definition 3.16.

Theorem 3.17. *Let $f : M \to M$ be an expanding map of a closed manifold M. Then there always exist a Riemannian metric $\| \cdot \|$ on M and a $\lambda > 1$ such that*

$$\|Df^n(v)\| \geq \lambda^n \|v\|$$

for every $n \in \mathbb{N}$.

As explained in Example 1.1, the easiest examples of an expanding map are given by the self-covers of the circle. We generalize this example to the class of infra-nilmanifolds.

Example 3.18. Let $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ be an affine infra-nilmanifold endomorphism. If $\bar{\alpha}$ only has eigenvalues of absolute value > 1, then $\bar{\alpha}$ is an expanding map and $\bar{\alpha}$ is called an expanding affine infra-nilmanifold endomorphism. We give some concrete examples of expanding affine infra-nilmanifold endomorphism.

(i) Let
$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \in \mathrm{GL}(n, \mathbb{Q})$$
be a diagonal matrix with $\lambda_i \in \mathbb{Z}$ for every i. If $|\lambda_i| > 1$ for every i, then A induces an expanding toral endomorphism on $\mathbb{T}^n$. In particular, every torus $\mathbb{T}^n$ has an expanding map.

(ii) Consider the nilmanifold $M = H_3(\mathbb{Z})\backslash H_3(\mathbb{R})$. Let $a, b \in \mathbb{Z}$ and take $\alpha : H_3(\mathbb{R}) \to H_3(\mathbb{R})$ the automorphism defined as
$$\alpha\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & ax & abz \\ 0 & 1 & by \\ 0 & 0 & 1 \end{pmatrix}.$$
Since $\alpha(H_3(\mathbb{Z})) \subseteq H_3(\mathbb{Z})$, the automorphism α induces an nilmanifold endomorphism $\bar{\alpha} : M \to M$. If $|a| > 1$ and $|b| > 1$, then $\bar{\alpha}$ is an expanding nilmanifold endomorphism.

The map α also induces an expanding nilmanifold endomorphism on every nilmanifold $N_k\backslash H_3(\mathbb{R})$.

In essentially the same way as Example 1.1 it can be shown that every expanding affine infra-nilmanifold endomorphism $\bar{\alpha}$ is chaotic.

It is harder to construct infra-nilmanifolds not admitting an expanding map. Such examples will be given in Chapter 6.

Properties

The dynamical properties of expanding maps are already described in the first paper [96] studying them. In this paper M. Shub shows that every manifold admitting an expanding diffeomorphism is diffeomorphic to $\mathbb{R}^n$ for some n. By studying the lifts of expanding maps to the universal cover, he proofs the following theorem.

Theorem 3.19 (Shub, 1969)**.** *Every expanding map of a closed Riemannian manifold has chaotic behavior.*

This generalizes Example 1.1 above.

In the same paper, he also shows that if two expanding maps on the same manifold are homotopic, they are topologically conjugate. This implies the following result.

Theorem 3.20 (Shub, 1969)**.** *Every expanding map of a closed Riemannian manifold is structurally stable.*

The only examples of expanding maps we gave above were expanding affine infra-nilmanifold endomorphisms. The following result by M. Gromov in [54] shows that these are in fact the only possible ones.

Theorem 3.21 (Gromov, 1981)**.** *Every expanding map on a closed manifold is topologically conjugate to an expanding affine infra-nilmanifold endomorphism.*

In fact, the major contribution of M. Gromov is showing that groups of polynomial growth are virtually nilpotent.

Theorem 3.22 (Gromov, 1981, Polynomial growth theorem)**.** *Let G be a finitely generated group. The group G is virtually nilpotent if and only if G has polynomial growth.*

One of the results in [50] is that the fundamental group of a closed manifold admitting an expanding map has polynomial growth. The fundamental group of a closed manifold is always finitely generated and torsion-free and therefore Theorem 3.22 implies that the fundamental group of such a manifold is isomorphic to an almost-Bieberbach group. The results of [63, 97] then show that Theorem 3.21 holds.

By Theorem 3.21, the problem of determining the infra-nilmanifolds admitting an expanding map is equivalent to determining the infra-nilmanifolds which have an expanding affine infra-nilmanifold endomorphism. An interesting question is then if we can give an algebraic way of describing the infra-nilmanifolds with an expanding infra-nilmanifold endomorphism.

Research question 3. Give an algebraic description of the infra-nilmanifolds supporting an expanding map.

Part II gives a complete answer to Research Question 3.

Expanding maps on infra-nilmanifolds

Although Theorem 3.21 already dates from 1981, there were not many results about the existence of expanding maps on infra-nilmanifolds before this thesis. We give a short overview.

A first type of results is finding properties of the infra-nilmanifolds which support an expanding map. In the paper [35] the authors show that the existence of an expanding map on an infra-nilmanifold $\Gamma\backslash G$ implies the existence of a positive grading on the Lie algebra $\mathfrak{g}$ corresponding to G.

Other results construct expanding maps on certain classes of infra-nilmanifolds. In [76] it is shown that every infra-nilmanifold modeled on a 2-step nilpotent Lie group admits an expanding map. This result was later generalized to the situation of homogeneous Lie algebras in [36]. A homogeneous Lie algebra is a Lie algebra $\mathfrak{g}$ with a positive grading

$$\mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{g}_i$$

such that $\mathfrak{g}$ is generated as Lie algebra by the subspace $\mathfrak{g}_1$.

Theorem 8.21, which is the main result of this dissertation about expanding maps, implies all these results.

There is another type of questions which follow naturally from Theorem 3.21. Let M be a manifold which is homeomorphic to an infra-nilmanifold supporting an expanding map, is it true that also M admits an expanding map? In the papers [47, 49] the authors construct examples of manifolds admitting an expanding map and which are homeomorphic but not diffeomorphic to a torus $\mathbb{T}^n$. The main idea is to start from a torus $\mathbb{T}^n$ and glue an exotic sphere to it such that the resulting manifold still admits an expanding map. The manifold constructed in this way are homeomorphic to the torus $\mathbb{T}^n$ you started from but have an exotic differentiable structure.

3.3 Anosov diffeomorphisms

Anosov diffeomorphisms on a manifold M also expand some vectors of TM, just as expanding maps, but there are some vectors which are contracted by the diffeomorphism. Since Anosov diffeomorphisms combine these two types of behavior, their structure is more complicated and therefore less is known about these dynamical systems.

What is an Anosov diffeomorphism?

The definition of an Anosov diffeomorphisms generalizes the idea of Arnold's cat map, as introduced in Example 1.2.

Definition 3.23. Let M be a closed manifold with Riemannian metric $\|\cdot\|$. A diffeomorphism $f : M \to M$ is called **Anosov** if there exist a continuous splitting of the tangent bundle $TM = E^s \oplus E^u$ such that

(1) the splitting $TM = E^s \oplus E^u$ is Df-invariant, i.e. the subbundles E^s and E^u are preserved under the map $Df : TM \to TM$;

(2) there exists constants $c > 0, \lambda > 1$ such that

$$\forall v \in E^u,\ \forall n \in \mathbb{N} : \|Df^n(v)\| \leq \frac{1}{c\lambda^n}\|v\|,$$

$$\forall v \in E^s,\ \forall n \in \mathbb{N} : \|Df^n(v)\| \geq c\lambda^n\|v\|.$$

By the same argument as in the case of expanding maps, Definition 3.23 does not depend on the choice of Riemannian metric on M. There always exists a Riemannian metric $\|\cdot\|$ such that the constant c can be chosen equal to 1 as well, see [85].

Theorem 3.24. *If $f : M \to M$ is an Anosov diffeomorphism on the closed manifold M with splitting $TM = E^s \oplus E^u$, then there exists a Riemannian metric $\|\cdot\|$ on M such that*

$$\forall v \in E^u,\ \forall n \in \mathbb{N} : \|Df^n(v)\| \geq \lambda^{-n}\|v\|,$$

$$\forall v \in E^s,\ \forall n \in \mathbb{N} : \|Df^n(v)\| \leq \lambda^n\|v\|.$$

Examples

The easiest example of an Anosov diffeomorphism, which we already mentioned in Chapter 1, is called Arnold's cat map.

This example generalizes easily to the class of infra-nilmanifolds.

Example 3.25. Let $\Gamma\backslash G$ be an infra-nilmanifold and $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ an affine infra-nilmanifold automorphism. The diffeomorphism $\bar{\alpha}$ is called a hyperbolic affine infra-nilmanifold automorphism if it has no eigenvalues of absolute value equal to 1. We show that in this case, $\bar{\alpha}$ is an Anosov diffeomorphism.

Consider the Lie algebra $\mathfrak{g} \simeq T_e G$ corresponding to G. Write $\mathfrak{g}$ as the direct sum $V^s \oplus V^u$ of subspaces V^s, V^u such that $\delta(V^s) = V^s$, $\delta(V^u) = V^u$ and δ only has eigenvalues < 1 in absolute value on V^s and > 1 on V^u. Define the vector bundles $\bar{E}^s$ and $\bar{E}^u$ on G by taking the left translates of the vector spaces V^s and V^u. Since every element of the holonomy group F commutes with some power of δ, these bundles are also invariant under F and thus there exist subbundles E^s and E^u covered by the bundles $\bar{E}^s$ and $\bar{E}^u$ on the infra-nilmanifold $\Gamma\backslash G$.

By construction the bundles $\bar{E}^s$ and $\bar{E}^u$ are invariant under δ and thus also under $(g, \delta) = \alpha$. This implies that the bundles E^s and E^u are invariant under the affine infra-nilmanifold endomorphism $\bar{\alpha}$. We leave the details for the reader to check.

We give some concrete examples of Anosov diffeomorphisms on infra-nilmanifolds constructed in this way.

(i) On $\mathbb{T}^2$ we had the example of Arnold's cat map, see Example 1.2. Consider the matrix
$$B = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
with $\det(B) = 1$, then a computation shows that B is hyperbolic. Thus B induces an Anosov diffeomorphism on the 3-dimensional torus $\mathbb{T}^3$. By combining the matrices A of Example 1.2 and B, it is possible to construct Anosov diffeomorphism on every n-dimensional torus $\mathbb{T}^n$ for $n > 1$.

The circle S^1 doesn't admit an Anosov diffeomorphism. Indeed, if $f : S^1 \to S^1$ would be an Anosov diffeomorphism, then the dimension of the bundle E^s or E^u of Definition 3.23 would be 0 and thus f or f^{-1} is an expanding diffemorphism. This is impossible because [96] shows that expanding diffeomorphisms only exists on spaces diffeomorphic to $\mathbb{R}^n$.

(ii) The nilmanifold $H_3(\mathbb{Z})\backslash H_3(\mathbb{R})$ does not admit an Anosov diffeomorphism. Let φ be an automorphism of $H_3(\mathbb{Z})$, then it also induces an automorphism on $Z(H_3(\mathbb{Z}))$ which is cyclic. This implies that φ has eigenvalue ± 1. Similarly also $N_k\backslash H_3(\mathbb{R})$ does not admit an Anosov diffeomorphism.

(iii) In the introduction we mentioned that the first non-toral example of an Anosov diffeomorphism was constructed by S. Smale in [99]. We present a simplified version of this construction underneath.

Consider the commutative ring $R = \mathbb{Z}[\sqrt{5}]$ and the non-trivial ring automorphism $\sigma : R \to R$ given by

$$\sigma(\sqrt{5}) = -\sqrt{5}$$

on the generator of R. Take the nilpotent group $N = H_3(\mathbb{Z}[\sqrt{5}])$, then σ induces an automorphism $\psi : N \to N$ given by

$$\psi\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & \sigma(x) & \sigma(z) \\ 0 & 1 & \sigma(y) \\ 0 & 0 & 1 \end{pmatrix}.$$

The group N does not form a discrete subgroup of the Lie group $H_3(\mathbb{R})$, therefore consider the injective group morphism

$$N \to H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$$

$$n \mapsto (n, \psi(n)),$$

then the image of N forms a discrete subgroup of the Lie group $G = H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$.

For every $r \in R$, there exists a group morphism $\alpha_r : N \to N$ be the group morphism given by

$$\alpha_r\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & rx & r^2 z \\ 0 & 1 & ry \\ 0 & 0 & 1 \end{pmatrix}.$$

The group morphisms α_r obviously satisfy the relation

$$\alpha_{r_1} \circ \alpha_{r_2} = \alpha_{r_1 r_2}$$

for every $r_1, r_2 \in R$. Every α_r uniquely extends to an automorphism of the Lie group G which maps the lattice N to itself and thus induces a nilmanifold endomorphism on $N\backslash G$. The eigenvalues of α_r are exactly $r, \sigma(r), r^2$ and $\sigma(r^2)$.

Consider the element $\lambda = 2 + \sqrt{5} \in \mathbb{Z}[\sqrt{5}]$ which satisfies $\lambda\sigma(\lambda) = -1$ and $|\lambda| > 1 > |\sigma(\lambda)|$. The group morphism α_λ is an automorphism since $\alpha_\lambda \circ \alpha_{-\sigma(\lambda)} = \mathbb{1}_N$. The automorphism α_λ is hyperbolic and thus induces an Anosov diffeomorphism on $N\backslash G$.

It is conjectured that, up to topologically conjugacy, the construction in Example 3.25 gives us every possible Anosov diffeomorphism on a closed manifold.

Conjecture 3.26. Every Anosov diffeomorphism of a closed manifold is topologically conjugate to a hyperbolic affine infra-nilmanifold automorphism.

The motivation for this conjecture is Theorem 3.21 for expanding maps and some partial results which are already known. This problem is already open for more than 45 years, ever since its original formulation in the paper [99] by S. Smale.

Partial results for Conjecture 3.26

We give a short overview of some partial results for Conjecture 3.26. Note that most of them are based on the paper [50] about Anosov diffeomorphisms. Recently it was pointed out by K. Dekimpe in [30] that there are some crucial errors in that paper. Therefore it is uncertain at the moment if the results are true, since it has to be checked if the proofs still work when using the notion of affine infra-nilmanifold endomorphisms instead of just infra-nilmanifold endomorphisms. Since we expect that most of the proofs can be generalized to affine infra-nilmanifold endomorphisms, we do state these results here.

The first partial answer for Conjecture 3.26 already describes the situation in low dimensions.

Theorem 3.27 (Newhouse, 1970)**.** *Let $f : M \to M$ be an Anosov diffeomorphism on a closed Riemannian manifold M. Let $TM = E^s \oplus E^u$ be the splitting of the tangent bundle in its stable part E^s and unstable part E^u. If $\dim(E^s) = 1$ or $\dim(E^u) = 1$, then f is topologically conjugate to a hyperbolic infra-nilmanifold automorphism on a torus.*

A proof of this theorem due to S.E. Newhouse can be found in [87]. An immediate consequence is that every manifold of dimension 3 or smaller which supports an Anosov diffeomorphism is homeomorphic to a torus. There was a generalization of this result in [62] to Anosov diffeomorphisms where $\dim(E^s) = 2$ or $\dim(E^u) = 2$, but this result is false, even on nilmanifolds, as we will show in Chapter 9.

The next result was given by A. Manning in [84] and solves the conjecture for Anosov diffeomorphisms on an infra-nilmanifold.

Theorem 3.28 (Manning, 1974)**.** *Let $f : \Gamma\backslash G \to \Gamma\backslash G$ be an Anosov diffeomorphism on an infra-nilmanifold. Then f is topologically conjugate to an affine infra-nilmanifold automorphism of $\Gamma\backslash G$.*

In a series of papers, see [11, 12, 13, 14, 15], M. Brin tackles the problem of Anosov diffeomorphism with pinched spectrum.

Theorem 3.29 (Brin, 1981)**.** *Let $f : M \to M$ be an Anosov diffeomorphism such that f has a pinched spectrum. Then f is topologically conjugate to a hyperbolic affine infra-nilmanifold automorphism.*

Roughly said, the condition of pinched spectrum means that the constants controlling the contraction and expansion can be chosen not too close to 1.

The key step in the papers [11, 12, 13, 14, 15] is the observation that the lifts of the stable and unstable leaves to the universal cover are polynomially distorted in the universal cover. This means that the distance function induced by the paths which stay in the leaves can be expressed as a polynomial of the distance induced by all possible paths in the universal cover.

In [59], A. Hammerlindl uses this condition of polynomial distortion of the stable and unstable leaves to define the notion of 'polynomial Global Product structure' of an Anosov diffeomorphism. The main result states that such a polynomial Global Product structure implies that that the Anosov diffeomorphism is topologically conjugate to a hyperbolic affine infra-nilmanifold endomorphism.

Theorem 3.30 (Hammerlindl, 2014)**.** *An Anosov diffeomorphism of a closed manifold is topologically conjugate to a hyperbolic affine infra-nilmanifold endomorphism if and only if it has polynomial Global Product structure.*

The arguments follow the proofs of [12] quite literally.

All these partial results, except the one of [87], have in common that they start from some knowledge of the fundamental group or a property which leads to some polynomial growth of the fundamental group. Indeed, one of the biggest problems for finding a proof of Conjecture 3.26 is that almost nothing is known about the fundamental group of manifolds supporting an Anosov diffeomorphism.

As explained above, the key step in the proof of Theorem 3.21 for expanding maps was the result relating the geometric property of polynomial growth (which follows easily from having an expanding map, see Theorem 11.9 giving a more general statement) to the algebraic property of being virtually nilpotent. A similar result relating the geometric conditions following from an Anosov diffeomorphism to some algebraic condition of the fundamental group is the missing link for Conjecture 3.26.

A positive answer to Conjecture 3.26 would imply that, up to homeomorphism, only infra-nilmanifolds admit an Anosov diffeomorphism. Not every infra-nilmanifold admits an Anosov diffeomorphism and having such a diffeomorphism puts a rather strong condition on the fundamental group of these infra-nilmanifolds. This motivates one of the main research questions of this thesis.

Research question 4. Are there algebraic ways of describing the infra-nilmanifolds supporting an Anosov diffeomorphism?

By Theorem 3.28 the existence of an Anosov diffeomorphism is equivalent to the existence of a hyperbolic affine infra-nilmanifold endomorphism. Although there are some doubts about this result due to the mistakes in [50], this last statement does hold because of [30].

Another important research type is to study manifolds admitting an Anosov diffeomorphism and which are homeomorphic but not diffeomorphic to an infra-nilmanifold. In the paper [48], the authors construct a big class of infra-nilmanifolds admitting an Anosov diffeomorphisms but which are not diffeomorphic to an infra-nilmanifold. The exact result is as follows.

Theorem 3.31 (Farrell - Gogolev, 2012)**.** *Let M be an infranilmanifold of dimension $n \geq 7$ which admits an Anosov diffeomorphism of codimension k commuting with an expanding map on M. Then the connected sum $M \# \Sigma$ with Σ a homotopy sphere from the Gromoll group Δ^n_{k+1} also admits a codimension k Anosov diffeomorphism.*

In Chapter 8 we will show that the condition in this result can be relaxed to the existence of an expanding map on M.

Properties

When the diffeomorphisms given by Definition 3.23 were first introduced, they were called U-diffeomorphisms. It was only later, after their first important property in [2] that S. Smale started calling them Anosov diffeomorphisms, in honorary of D. Anosov.

Theorem 3.32 (Anosov, 1967)**.** *Every Anosov diffeomorphism of a closed manifold is structurally stable.*

In Example 1.2 we showed that Arnold's cat map, which is the prototype of an Anosov diffeomorphism, is a chaotic dynamical systems. The same proof shows in fact that every hyperbolic affine infra-nilmanifold endomorphism is chaotic which form all known examples of Anosov diffeomorphism up to topological conjugacy. Although all known examples of Anosov diffeomorphism are chaotic, it is still an open question if this is true in general.

Conjecture 3.33. Every Anosov diffeomorphism on a closed manifold is chaotic.

Of course every Anosov diffeomorphism has sensitive dependence on initial conditions, in the same way as it holds for Example 1.2. So to prove that every Anosov diffeomorphism is chaotic, it suffices to show that the periodic points are dense and that it is topologically transitive.

We mention some partial results about chaotic behavior in relation to the non-wandering set.

Theorem 3.34. *For every Anosov diffeomorphism $f : M \to M$ the periodic points* $\mathrm{Per}(f)$ *forms a dense subset of* $\Omega(f)$*.*

Theorem 3.35 (Topological decomposition theorem)**.** *Let $f : M \to M$ be an Anosov diffeomorphism, then the non-wandering set $\Omega(f)$ can be written as the disjoint union*

$$\Omega(f) = \bigcup_{i=1}^{k} B_i$$

of f-invariant closed subsets B_i such that $f|_{B_i}$ is topologically transitive.

So both theorems imply that it suffices to show that $\Omega(f) = M$ for every Anosov diffeomorphism.

3.4 Anosov diffeomorphisms on infra-nilmanifolds

In this section, we summarize the known results about the existence of Anosov diffeomorphisms on infra-nilmanifolds. Since every Anosov diffeomorphism f on an infra-nilmanifold $\Gamma\backslash G$ has a lift to the nilmanifold $N\backslash G$ which is also an Anosov diffeomorphism, the first step is to understand which nilmanifolds admit an Anosov diffeomorphism.

The following result of K. Dekimpe and K. Verheyen in [38] already gives a first algebraic description of the infra-nilmanifolds admitting an Anosov diffeomorphism depending only on the rational holonomy representation.

Theorem 3.36. *Let $\Gamma\backslash G$ be an infra-nilmanifold with associated rational holonomy representation $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$. Then $\Gamma\backslash G$ admits an Anosov diffeomorphism if and only if there exists a hyperbolic automorphism $\varphi \in \mathrm{Aut}(N^{\mathbb{Q}})$ which commutes with every element of $\rho(F)$ and such that φ is integer-like.*

Again this shows that the first step is to understand the nilpotent Lie algebras $\mathfrak{n}^{\mathbb{Q}}$ admitting a hyperbolic integer-like automorphism. The next step is then to study finite subgroups of automorphisms of this Lie algebra and the automorphisms commuting with this finite subgroup.

Anosov diffeomorphisms on nilmanifolds

There are a few classes of nilmanifolds on which the existence of an Anosov diffeomorphism is completely characterized. Examples of such classes include nilmanifolds modeled on free nilpotent Lie groups, on Lie groups associated to graphs and on Lie groups of dimension ≤ 8.

Theorem 3.37. *Let M be a nilmanifold modeled on a free nilpotent Lie group $G_{n,c}$ on n generators of nilpotency class c. Then M admits an Anosov diffeomorphism if and only if $n > c$.*

For $c = 1$, this gives us the result on tori as explained in Example 3.25. The case $c = 2$ was treated in the paper [37] and the general version was given in [23]. A constructive proof of this result is given by K. Dekimpe and K. Verheyen in [38].

A second class of nilmanifolds which is described are the ones modeled on Lie groups associated to graphs. These Lie groups were introduced in [22]. A more detailed introduction is given in the Appendix. We state the result of [22] which gives a complete algebraic description of the nilmanifolds associated to graphs admitting an Anosov diffeomorphism.

Theorem 3.38. *Let $X = (V, E)$ be a finite graph and $N_X \backslash G_X$ the nilmanifold associated to the graph X. The nilmanifold $N_X \backslash G_X$ admits an Anosov diffeomorphism if and only if for every $v \in V$, the maximal coherent subset of X containing v has at least three elements or is equal to $\{v, w\}$ with $vw \notin E$.*

Let $\mathfrak{n}^{\mathbb{Q}}_{n,2}$ the free nilpotent Lie algebra of nilpotency class 2 over $\mathbb{Q}$. The nilmanifolds corresponding to 2-step nilpotent rational Lie algebras given by a quotient ${}^{\mathfrak{n}^{\mathbb{Q}}_{n,2}}\!/_{L}$ where L is one-dimensional are given in [33]. The nilmanifolds admits an Anosov diffeomorphism if and only if $n \geq 5$ and L is generated by an element of the form $[X, Y]$ with $X, Y \in \mathfrak{n}^{\mathbb{Q}}_{n,2}$.

Finally, there is a classification of the nilmanifolds admitting an Anosov diffeomorphism up to dimension 9. The classification was realized in different steps throughout the years. A first result due to W. Malfait in [82] and C. Cassidy, N. Kennedy and D. Scevenels in [19] classifies the nilmanifolds up to dimension 6. Later, this classification was simplified (using the relation with nilpotent Lie algebras) in [73, 75] and extended up to dimension 8. Finally the case of dimension 9 was added recently by [80].

The other results about Anosov diffeomorphisms on nilmanifolds try to construct examples which specific properties. Most of these examples are constructed explicitly, starting from a Lie algebra given by generators and relations and then a rational form is constructed which admits a hyperbolic integer-like automorphism. We give more details about these construction and a generalization avoiding all computations in Chapter 9.

The following result from J. Lauret, see [72], shows that a complete classification of the nilmanifolds admitting an Anosov diffeomorphism is hard to realize.

Theorem 3.39. *Let $\Gamma\backslash G$ be a nilmanifold modeled on a Lie group G such that the Lie algebra $\mathfrak{g}$ corresponding to G has a positive grading. Then there exists a nilmanifold modeled on $G \oplus G$ which admits an Anosov diffeomorphism.*

Since the class of rational Lie algebras with positive grading is not yet classified and we can construct pairwise non-isomorphic nilmanifolds from these Lie algebras, the classification of nilmanifolds admitting an Anosov diffeomorphism is not doable at the moment.

Anosov diffeomorphisms on infra-nilmanifolds

The first result about Anosov diffeomorphisms on a class of infra-nilmanifolds was given by H. Porteus in [90]. This result classifies all flat manifolds, so infra-nilmanifolds modeled on $\mathbb{R}^n$ which admit an Anosov diffeomorphism.

Theorem 3.40 (Porteous, 1972)**.** *Let $\Gamma\backslash\mathbb{R}^n$ be a flat manifold with holonomy representation $\rho : F \to \mathrm{GL}(n,\mathbb{Z})$. The manifold $\Gamma\backslash\mathbb{R}^n$ admits an Anosov diffeomorphism if and only if every $\mathbb{Q}$-irreducible representation of ρ which occurs with multiplicity 1 splits over the reals $\mathbb{R}$.*

This statement was later generalized to infra-nilmanifolds modeled on a free nilpotent Lie group and with an abelian holonomy group in [39].

Theorem 3.41 (Dekimpe-Verheyen, 2013)**.** *Let $\Gamma\backslash G$ be an infra-nilmanifold modeled on a free c-step nilpotent Lie group G with abelian holonomy group F. Consider the associated abelianized rational holonomy representation $\overline{\varphi} : F \to \mathrm{Aut}\left(N_{\mathbb{Q}}/[N_{\mathbb{Q}}, N_{\mathbb{Q}}]\right)$ where F is the holonomy group of Γ. Then the following statements are equivalent:*

$$\Gamma\backslash G \text{ admits an Anosov diffeomorphism.}$$

$$\Updownarrow$$

Every $\mathbb{Q}$-irreducible component of $\bar{\varphi}$ that occurs with multiplicity m, splits in more than $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

For $c = 1$, the free nilpotent Lie group $G \simeq \mathbb{R}^n$ and thus this result does generalizes the theorem of Porteous in the case of abelian holonomy groups.

Finally we also mention the paper [83] of W. Malfait containing some results about Anosov diffeomorphism on infra-nilmanifolds of dimension 6. In this paper, he studies the possible abelianized rational holonomy representations in these cases and shows that some of them do not occur. The work in this paper is far from complete, since for the remaining cases it is not known if there exists a hyperbolic and integer-like automorphism commuting with the representation nor if there exists an infra-nilmanifold with these holonomy groups. Since even this questions are still open, a classification of infra-nilmanifolds of dimension ≤ 6 is still open.

3.5 Topological generalizations

Definitions 3.16 and 3.23 of expanding maps and Anosov diffeomorphisms on a manifold M are strongly based on the differentiable structure of M. Both definitions start from a Riemannian metric $\|\cdot\|$ and state how the length of tangent vectors $v \in TM$ changes under the derivative of the map. In this section we generalize the definitions to arbitrary metric spaces such that most techniques used to study them are still applicable to these generalizations.

Positively expansive maps

Most proofs concerning expanding maps only use the observation that such a map locally increases the distance between points of the manifold. An immediate consequence is that two points of the manifold which lie close to each other eventually move far apart under the expanding map. This idea motivates the following generalization of expanding maps to arbitrary metric spaces.

Definition 3.42. Let $f : X \to X$ be a continuous surjection of a metrix space X. We say that f is **positively expansive** if there exists some constant $e > 0$ such that for every $x, y \in X$ it holds that

$$d(f^n(x), f^n(y)) < e \;\; \forall n \in \mathbb{N} \Rightarrow x = y.$$

Recall that if (X, d) is a metric space, then we call a metric d' on X compatible with d if they induce the same topology on X. If X is a compact metric space, the definition of positively expansive map does not depend on the choice of metric of X, as long as the metrics are compatible.

Since the definition of positively expansive maps is completely in terms of the metric, we have the following result.

Theorem 3.43. *Let $f : X \to X$ and $g : Y \to Y$ be two surjections of metric spaces X and Y which are topological conjugate. Then f is positively expansive if and only if g is positively expansive.*

Note that this result is not true for expanding maps, so Definition 3.42 is more flexible to work with under topological conjugation.

An important result of W. Reddy in [91] states there exists a metric on X such that positively expansive maps indeed locally increase distances.

Theorem 3.44 (Reddy, 1982)**.** *Let $f : X \to X$ be a positive expansive map on a compact metric space X. There exist constants $\delta > 0, \lambda > 1$ and a metric d on X, compatible with the original metric, such that for all $x, y \in X$, we have that*

$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) > \lambda d(x, y).$$

By essentially the same proof as in the case of expanding maps, it was shown in [61] that every positively expansive map on a closed manifold is topologically conjugate to an expanding infra-nilmanifold endomorphism.

Theorem 3.45 (Hiraide, 1988)**.** *Every positively expansive map on a closed manifold is topologically conjugate to an expanding affine infra-nilmanifold endomorphism.*

The proof also works for topological manifolds and seems to generalize to even bigger classes of metric spaces which are sufficiently nice. Theorem 3.45 implies that the results in this thesis concerning expanding maps can be reformulated in terms of positively expansive maps. Since the name expanding maps is used more frequently in literature, we keep using this name.

TA homeomorphisms

Just as in the case of expanding maps, there is a natural topological generalization of Anosov diffeomorphisms which avoids using the tangent bundle of a manifold. We give a short introduction based on [3].

Let $f : X \to X$ be a homeomorphism of a metrix space X. The map f is expansive if there exists some $e > 0$ such that if for $x, y \in X$ arbitrary points we have that

$$d(f^n(x), f^n(y)) < e \ \ \forall n \in \mathbb{Z} \to x = y.$$

The difference with positive expansive maps is that also the inverse of f is used this time, so expansive is a weaker property than positively expansive. The intuition is that different points of our space eventually move away from each other by moving forward or backward under the map f.

The homeomorphism f has the pseudo orbit tracing property, abbreviated as POTP, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $(x_i)_{i\in\mathbb{Z}}$ is a sequence of points such that $d(f(x_i), x_{i+1}) < \delta$, there exists some $x \in X$ such that

$$d(f^i(x), x_i) < \epsilon.$$

So every sequence of points which is almost an orbit lies close to an actual orbit of the map f.

Definition 3.46. A topological Anosov homeomorphism (**TA homeomorphism**) is a homeomorphism $f : X \to X$ of a compact metric space X which is expansive and has POTP.

Every Anosov diffeomorphism is expansive and has the POTP, see [3].

Starting from the definiton of a TA homeomorphism, one can show that the homeomorphism f can be locally written as the direct product of a contracting homeomorphism and an expanding homeomorphism. For this, we first introduce the stable set of a point $x \in X$ as

$$W^s(x, d) = \{y \in X \mid d(f^n(x), f^n(y)) \to 0\}$$

and the unstable set $W^u(x, d)$ which is the stable set of x for the map f^{-1}. The local stable set $W^s_\epsilon(x, d)$ for $\epsilon > 0$ is then equal to

$$W^s_\epsilon(x, d) = W^s(x, d) \cap B(x, \epsilon)$$

where $B(x, \epsilon)$ is the ball of radius ϵ around x and similarly we define the local unstable set $W^u_\epsilon(x, d)$.

Theorem 3.47 (Reddy, 1983)**.** *Let $f : X \to X$ be an expansive homeomorphism, then there exists constants $\delta > 0, 0 < \lambda < 1$ and a compatible metric d on X such that*

$$\forall y \in W^s_\epsilon(x, d) : d(f^n(x), f^n(y)) \le c\lambda^n d(x, y)$$

$$\forall y \in W^u_\epsilon(x, d) : d(f^{-n}(x), f^{-n}(y)) \le c\lambda^n d(x, y)$$

for all $n \in \mathbb{N}$.

Therefore TA homeomorphisms are the good generalization of Anosov diffeomorphisms to the topological setting. In fact, most techniques for studying Anosov diffeomorphism can be directly generalized to TA homeomorphisms.

The advantage of working with TA homeomorphisms is that the definition is completely in terms of the metric on X.

Theorem 3.48. *Let $f : X \to X$ and $g : Y \to Y$ be two homeomorphisms of metric spaces X and Y which are topological conjugate. Then f is a TA homeomorphism if and only if g is a TA homeomorphism.*

For Anosov diffeomorphisms this is not true; there are diffeomorphisms topologically conjugate to an Anosov diffeomorphisms which are itself not Anosov. An example is given in [3, Theorem 9.1.1.].

We conjecture that every topological manifold admitting a TA homeomorphism is homeomorphic to an infra-nilmanifold.

Conjecture 3.49. Let $f : X \to X$ be a TA homeomorphism of a topological manifold X. Then f is topologically conjugate to a hyperbolic affine infra-nilmanifold automorphism.

Although the following is not written down anywhere explicitly, the existence of TA homeomorphisms on infra-nilmanifolds is equivalent to the existence of Anosov diffeomorphisms.

Theorem 3.50. *Let $\Gamma\backslash G$ be an infra-nilmanifold, then $\Gamma\backslash G$ admits an Anosov diffeomorphism if and only if $\Gamma\backslash G$ admits a TA homeomorphism.*

So the results about Anosov diffeomorphisms in this thesis could also be stated in terms of TA homeomorphisms. Since Anosov diffeomorphism are the most familiar name in dynamical systems, we stick to this name.

If the proof of A. Manning for Theorem 3.28 can be repaired, the same proof will also give us the following.

Conjecture 3.51. Let $\Gamma\backslash G$ be an infra-nilmanifold and $f : \Gamma\backslash G \to \Gamma\backslash G$ a TA homeomorphism. Then f is topologically conjugate to a hyperbolic affine infra-nilmanifold automorphism.

Chapter 4

Hyperbolic algebraic units in number fields

Let $\bar{\alpha}$ be an affine infra-nilmanifold endomorphism induced by the affine transformation $\alpha \in \operatorname{Aff}(G)$ on the infra-nilmanifold $\Gamma \backslash G$. Chapter 3 demonstrated that the dynamical properties of the self-map $\bar{\alpha}$ depend on its eigenvalues, which are defined as the eigenvalues of the linear part of α. For example if every eigenvalue λ of $\bar{\alpha}$ satisfies $|\lambda| > 1$, then $\bar{\alpha}$ is an expanding map. Similarly, if $\bar{\alpha}$ is an affine infra-nilmanifold automorphism with only hyperbolic eigenvalues, so of absolute value $\neq 1$, then $\bar{\alpha}$ is an Anosov diffeomorphism.

This chapter studies the properties of these eigenvalues of affine infra-nilmanifold endomorphisms in combination with the fields of which they are elements. Every eigenvalue is an algebraic integer and the smallest field containing them is a number field. If $\bar{\alpha}$ is moreover a diffeomorphism then every eigenvalue is an algebraic unit of this number field. To investigate the existence of certain affine infra-nilmanifold endomorphisms it is therefore important to study these algebraic units as a subject of their own.

The results of this chapter are relevant in two different ways through this thesis. On the one hand, the methods of algebraic number fields are the key ingredient for establishing the relation between gradings on Lie algebras and the existence of expanding maps in Chapter 8. On the other hand, the construction of hyperbolic algebraic units in number fields is an essential tool for producing Anosov diffeomorphisms both in Chapter 9 and 10.

This chapter starts by recalling the necessary background on number fields and

Galois extensions. Next, it gives two different techniques for finding hyperbolic algebraic units in number fields satisfying additional properties. Finally, the last section describes a method for constructing Galois extensions of a given degree for any number field.

4.1 Background on number theory

This first section introduces the standard definitions about algebraic units in number fields and fixes notations for this dissertation. A more detailed introduction is available in the standard references [100, 101].

Field extensions

Let E be any field and denote by $E^\times$ the multiplicative group of units in the field E. A field extension of E is a field F which contains E as a subfield, so $E \subseteq F$. The field F is in a natural way a vector space over the field E. The degree of the field extension F is defined as the dimension of the vector space F over E and denoted as $[F : E]$. If $E \subseteq K \subseteq F$ are field extension of finite degree, then the degrees satisfy the formula

$$[F : E] = [F : K][K : E]$$

which is similar to the formula for finite index subgroups.

For every $\theta_1, \ldots, \theta_k \in F$, we denote by $E(\theta_1, \ldots, \theta_n)$ the smallest field extension of E such that every $\theta_i \in E(\theta_1, \ldots, \theta_k)$. If E is a field of characteristic 0, then every field extension $E \subseteq F$ of finite degree is simple, meaning that there exists $\theta \in F$ such that $F = E(\theta)$. All the fields in this chapter and also in the remaining part of this dissertation are assumed to be of characteristic 0. There is a rich theory for fields of characteristic $p > 0$ as well, but because of our interest for self-maps on infra-nilmanifolds, we restrict our attention to these fields.

Let $F \supseteq E$ be a field extension of finite degree. Every $\alpha \in F$ is algebraic over E and thus the root of some nonzero polynomial $p(X) \in E[X]$. The minimal polynomial of α over E is the unique monic polynomial $p(X) \in E[X]$ of minimal degree such that α is a root of this polynomial. Every minimal polynomial is irreducible over the field E. If $p(X) \in E[X]$ is the minimal polynomial of $\alpha \in F$, then the field

$$E[X] \big/ \big(p(X)\big) \simeq E(\alpha)$$

and thus the degree of the minimal polynomial is equal to the degree of the field extension $[E(\alpha) : E]$.

Let $F \supseteq E$ be a field extension of degree n, then the Galois group of this field extension is defined as the group

$$\mathrm{Gal}(F, E) = \{\sigma : F \to F \mid \sigma \text{ is a field automorphism, } \sigma|_E = \mathbb{1}_E\}\,.$$

The order of the group $\mathrm{Gal}(F, E)$ is always smaller than or equal to n and we call F a Galois extension of E if the order of $\mathrm{Gal}(F, E)$ is exactly n. The following proposition gives an equivalent way of describing the Galois extensions.

Proposition 4.1. *A field F is a Galois extension of a field E if and only if F is the splitting field of a polynomial over E.*

Recall that F is the splitting field of a polynomial $p(X) \in E[X]$ if and only p considered as a polynomial over F splits in linear factors

$$p(X) = \prod_{i=1}^{n} (X - \theta_i)$$

with $\theta_i \in F$ and $F = E(\theta_1, \ldots, \theta_n)$. If $F \supseteq E$ is a field extension of finite degree, then there always exists a field extension $F' \supseteq F$ of finite degree such that F' is Galois over E.

To describe the fundamental result of Galois extensions, we first introduce some notations. Let $H \leq \mathrm{Gal}(F, E)$ be a subgroup, then we denote by F^H the subfield of F fixed by the group H, so

$$F^H = \{x \in F \mid \sigma(x) = x \;\; \forall \sigma \in H\}\,.$$

If K is a field such that $E \subseteq K \subseteq F$, then $\mathrm{Gal}(F, K)$ is a subgroup of the group $\mathrm{Gal}(F, E)$.

The following result is the fundamental theorem of Galois theory and describes the relation between subgroups of $\mathrm{Gal}(F, E)$ and subfields of E.

Theorem 4.2 (Galois correspondence)**.** *Let $F \supseteq E$ be a Galois extension, then the maps*

$$\begin{tikzcd} \{H \mid H \leq \mathrm{Gal}(F, E)\} \arrow[d, shift left, "F^{-}"] \\ \{K \mid E \subseteq K \subseteq F\}\,. \arrow[u, shift left, "\mathrm{Gal}(F,-)"] \end{tikzcd}$$

are mutual inverse bijections, where $F^{-}(H) = F^H$ *and* $\mathrm{Gal}(F,-)(K) = \mathrm{Gal}(F,K)$ *are the maps as introduced above the theorem. These bijections satisfy the following properties.*

(1) *Both maps are order-reversing.*

(2) *Normal subgroups correspond to the fields* K *which are Galois extensions of* E*. In this case, we have an isomorphism*

$$\mathrm{Gal}(K,E) \simeq {}^{\mathrm{Gal}(F,E)}\!\big/_{\mathrm{Gal}(F,K)}.$$

(3) *The index of the subgroup* $H \leq \mathrm{Gal}(F,E)$ *is equal to the degree of the field extension* $F^H \supseteq E$*, so*

$$[\mathrm{Gal}(F,E) : H] = [F^H : E].$$

Example 4.3. The subfield $E \subseteq F$ corresponds to the full subgroup $\mathrm{Gal}(F,E)$ under the Galois correspondence and thus $E = F^{\mathrm{Gal}(F,E)}$. In particular, for every $x \in F$ it holds that if $\sigma(x) = x$ for all $\sigma \in \mathrm{Gal}(F,E)$, then $x \in E$. This gives an easy way of showing that elements of the field F lie in the field E.

Number fields

A number field E is a field of characteristic 0 which has finite degree over its prime subfield $\mathbb{Q} \subseteq E$. Since E is a simple extension of $\mathbb{Q}$ in that case, there always exists $\theta \in E$ such that $E = \mathbb{Q}(\theta)$. This implies that the field E is isomorphic to a subfield of the complex numbers $\mathbb{C}$ and during this dissertation we always assume that number fields are given as a subfield of $\mathbb{C}$. We call a number field real if it is embedded as a subfield of $\mathbb{R} \subseteq \mathbb{C}$.

If α is an element of an algebraic number field E, then we call α an algebraic integer if the minimal polynomial of α over $\mathbb{Q}$ has coefficients in $\mathbb{Z}$. The set of algebraic integers of a number field E is a subring of E which we denote as $\mathcal{O}_E$. The unit group of the ring $\mathcal{O}_E$ is denoted as U_E and its elements are called algebraic units. Equivalently, an element $x \neq 0$ is an algebraic unit if and only if both x and x^{-1} are algebraic integer. Note that the definition of algebraic integer and therefore also of algebraic unit is independent of the number field E we are working in.

Example 4.4. The field $\mathbb{Q}$ is a number field of extension degree 1 over $\mathbb{Q}$. For every $q \in \mathbb{Q}$, the minimal polynomial is given by $p_q(X) = X - q \in \mathbb{Q}[X]$ and thus the algebraic integers of $\mathbb{Q}$ are exactly $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$. The unit group in $\mathbb{Z}$ is equal to $U_E = \{\pm 1\}$. This example is important to describe some properties of algebraic integers and algebraic units further on.

Let $E = \mathbb{Q}(\theta)$ be a number field of degree n over $\mathbb{Q}$ and $p_\theta(X) \in \mathbb{Q}[X]$ the minimal polynomial of θ. The degree of the polynomial p_θ is equal to n and denote by $\theta_1 = \theta, \ldots, \theta_n$ the n distinct roots of this polynomial. If $\sigma : E \to \mathbb{C}$ is a field monomorphism, then

$$p_\theta(\sigma(\theta)) = \sigma\left(p_\theta(\theta)\right) = \sigma(0) = 0$$

and thus $\sigma(\theta) = \theta_i$ for some $1 \leq i \leq n$. Since the monomorphism σ is completely determined by the image of θ, there are at most n possible monomorphisms $\sigma : E \to \mathbb{C}$.

Furthermore, for every $1 \leq i \leq n$, there exists a unique monomorphism $\sigma_i : E \to \mathbb{C}$ such that $\sigma_i(\theta) = \theta_i$. So there are exactly n different monomorphisms $\sigma : E \to \mathbb{C}$ and we write these monomorphisms as $\sigma_i : E \to \mathbb{C}$ for $1 \leq i \leq n$. If E is a Galois extension of $\mathbb{Q}$, then the monomorphisms σ_i are exactly the elements of the Galois group $\mathrm{Gal}(E, \mathbb{Q})$.

If the monomorphism σ_i satisfies $\sigma_i(E) \subseteq \mathbb{R}$, we call the monomorphism σ_i real, otherwise σ_i is called complex. The complex monomorphisms come in pairs, since if $\sigma_i : E \to \mathbb{C}$ is a complex monomorphism, the map $\bar{\sigma}_i : E \to \mathbb{C}$ given by

$$\bar{\sigma}_i(\alpha) = \overline{\sigma_i(\alpha)} \ \ \forall \alpha \in E$$

is a monomorphism different from σ_i. Hence the degree n satisfies $n = s + 2t$ where s is the number of real monomorphisms and $2t$ is the number of complex monomorphisms.

For every $\alpha \in E$ in a number field E, we define the E-conjugates of α as the complex numbers $\sigma_i(\alpha)$. If α is an algebraic integer, then also $\sigma_i(\alpha)$ is an algebraic integer in the field $\sigma_i(E)$ since it has the same minimal polynomial over $\mathbb{Q}$. This means that $\sigma_i(\mathcal{O}_E) = \mathcal{O}_{\sigma_i(E)}$. Let $\alpha \in U_E$, meaning that $\alpha^{-1} \in \mathcal{O}_E$, then it holds we have that $\sigma_i(\alpha^{-1}) \in \mathcal{O}_{\sigma_i(E)}$ and thus $\sigma_i(\alpha)$ is an algebraic unit in $\sigma_i(E)$.

The norm of α, denoted as $N_E(\alpha)$, is the product of all its E-conjugates, so

$$N_E(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha).$$

This gives us a map $N_E : E \to \mathbb{Q}$ which preserves the product, i.e. $N_E(\alpha\beta) = N_E(\alpha)N_E(\beta)$ for all $\alpha, \beta \in E$. If $\alpha, \beta \in E$ are E-conjugate, then $N_E(\alpha) = N_E(\beta)$ by definition. Let $\alpha \in \mathcal{O}_E$ be an algebraic integer, then $N_E(\alpha)$ is also an algebraic integer as it is the product of algebraic integers. Since $N_E(\alpha) \in \mathbb{Q}$, Example 4.4 implies that $N_E(\alpha) \in \mathbb{Z}$. Moreover, if α is an algebraic unit, then also $N_E(\alpha)$ is an algebraic unit and thus $N_E(\alpha) = \pm 1$. We conclude that the minimal polynomial of an algebraic unit of E has constant term ± 1, since the constant term is up to sign just the product of all the E-conjugates of the element. Vice versa, if an algebraic integer $\alpha \in \mathcal{O}_E$ satisfies $N_E(\alpha) = \pm 1$, then $\alpha \in U_E$ by the same argument.

Algebraic units

The importance of algebraic units in number fields lies in the study of affine infra-nilmanifold automorphisms. If $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ is an affine infra-nilmanifold automorphism, then $\bar{\alpha}$ induces an automorphism on the Fitting subgroup N of Γ. Hence its determinant is equal to ± 1 by Proposition 2.30 and therefore every eigenvalue of $\bar{\alpha}$ is an algebraic unit.

We show that the converse of this statement is true as well for nilmanifold endomorphisms, even when we only consider the eigenvalues induced on the abelianization $\left(N^{\mathbb{Q}}\right)^{\mathrm{ab}} = N^{\mathbb{Q}} / [N^{\mathbb{Q}}, N^{\mathbb{Q}}]$.

Proposition 4.5. *Let $\delta : N^{\mathbb{Q}} \to N^{\mathbb{Q}}$ be an automorphism such that the induced map on the abelian group $N^{\mathbb{Q}} / \gamma_2(N^{\mathbb{Q}})$ has only algebraic units as eigenvalues. Then* $\det(\delta) = \pm 1$.

Proof. The eigenvalues of δ are k-fold products of the eigenvalues of the induced map on the abelianization and are thus algebraic units. Since the characteristic polynomial of δ only has algebraic units as roots, this implies the statement. $\square$

By Proposition 2.30, having $\det(\delta) = \pm 1$ is equivalent to being an automorphism of an $\mathcal{F}$-group N.

For affine infra-nilmanifold endomorphisms $\bar{\alpha}$ induced by $\alpha \in \mathrm{Aff}(G)$, it holds that $\bar{\alpha}$ is a diffeomorphism if and only if the linear part of α has determinant ± 1, see Corollary 6.23. So Proposition 4.5 can also be applied to show that affine infra-nilmanifold endomorphisms are diffeomorphisms.

The group structure of the algebraic units U_E of a number field E is well-known because of Dirichlet Units Theorem.

Dirichlet's Unit Theorem. The group of units U_E of a number field E of degree n is isomorphic to

$$W \oplus \mathbb{Z}^{s+t-1}$$

where $W \leq U_E$ is the finite subgroup of roots of unity in U_E and $n = s + 2t$ with s the number of real monomorphisms of E.

We will explain in more detail how the proof of Dirichlet Units Theorem works, since this will give us information about the hyperbolic units in E.

Consider the map

$$\begin{aligned} l_E : E^\times \to \mathbb{R}^{s+t} : \\ \mu \mapsto \left(\log|\sigma_1(\mu)|, \ldots, \log|\sigma_s(\mu)|, \log|\sigma_{s+1}(\mu)|, \ldots, \log|\sigma_{s+t}(\mu)|\right). \end{aligned} \tag{4.1}$$

The map l_E satisfies

$$l_E(\alpha\beta) = l_E(\alpha) + l_E(\beta)$$

for every $\alpha, \beta \in E^\times$, so it induces a group morphisms between the groups $E^\times$ and $\mathbb{R}^{s+t}$. The subgroup $U_E \leq E^\times$ of algebraic units is mapped to the subspace $V \subseteq \mathbb{R}^{s+t}$ given by the equation

$$V = \left\{ (x_1, \ldots, x_{s+t}) \in \mathbb{R}^{s+t} \,\middle|\, \sum_{i=1}^{s} x_i + \sum_{j=s+1}^{t} 2x_j = 0 \right\}$$

and every element $\alpha \in \mathcal{O}_E$ with $l_E(\alpha) \in V$ is an algebraic unit. We denote by $l : U_E \to \mathbb{R}^{s+t}$ the restriction of l_E to the subgroup U_E.

Every element in the kernel only has E-conjugates of absolute value 1. The following lemma describes how such elements look like.

Lemma 4.6. *Let $p(X) \in \mathbb{Q}[X]$ a polynomial which only has roots of absolute value 1, then every root of $p(X)$ is a root of unity.*

Thus every element in the kernel of l is a root of unity as described in the following proposition.

Proposition 4.7. *The kernel* $\mathrm{Ker}(l)$ *is exactly the (finite) subgroup of roots of unity in E.*

The image of the map l forms a discrete subset of the subspace V and in facts forms a uniform lattice by the following result.

Proposition 4.8. *The image of l is a uniform lattice of the subspace V.*

Since U_E is an abelian group, both lemmas imply Dirichlet's unit theorem via the first isomorphism theorem.

4.2 Hyperbolic algebraic units

Every Anosov diffeomorphism induces an automorphism on the fundamental group and thus its eigenvalues are hyperbolic algebraic units. In this section, we discuss the existence of such hyperbolic algebraic units in number fields which satisfy some additional properties. The main idea is to adapt the proof of Dirichlet Units Theorem and translating the additional properties in some algebraic or topological condition.

During this section, E is a number field of degree n over $\mathbb{Q}$ and just as before we write $\sigma_i : E \to \mathbb{C}$ with $1 \leq i \leq n$ for the n distinct monomorphisms.

The first interesting property for hyperbolic units is to be c-hyperbolic, which plays an important role when studying Anosov diffeomorphisms on free nilpotent Lie groups as in Chapter 10.

Definition 4.9. An algebraic unit $\lambda \in U_E$ is called **c-hyperbolic** if

$$\forall k \in \{1, \ldots, c\}, \;\; \forall\, i_1, i_2, \ldots, i_k \in \{1, \ldots, n\} :$$

$$|\sigma_{i_1}(\lambda) \ldots \sigma_{i_k}(\lambda)| \neq 1.$$

The notion of a c-hyperbolic unit is the translation of c-hyperbolic integer-like matrices as introduced in [39] to number fields. A matrix $A \in \mathrm{GL}(n, \mathbb{Q})$ with characteristic polynomial in $\mathbb{Z}[X]$ and determinant ± 1 is called integer-like. By definition an element of E is c-hyperbolic if and only if the companion matrix $L(p)$ of its minimal polynomial $p(X)$ over $\mathbb{Q}$ is c-hyperbolic and integer-like.

There is an obvious condition on number fields such that it does not have c-hyperbolic algebraic units. Let $\alpha \in U_E$, then $N_E(\alpha) = \pm 1$ since α is an algebraic unit. By definition we have that

$$N_E(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha)$$

is the product of n conjugates of α. This implies that E cannot have n-hyperbolic algebraic units.

If we assume moreover that E is totally imaginary, the monomorphisms can be written as $\sigma_1, \ldots, \sigma_t, \bar{\sigma}_1, \ldots, \bar{\sigma}_t$ where $t = \frac{n}{2}$. This implies that

$$|\prod_{i=1}^{t} \sigma_i(\alpha)|^2 = |N_E(\alpha)| = 1$$

and thus also

$$|\prod_{i=1}^{t}\sigma_i(\alpha)| = 1.$$

This implies that E in this case doesn't have algebraic units which are $\frac{n}{2}$-hyperbolic.

Another desirable property for hyperbolic algebraic units to have is to be a Pisot number. We call a real algebraic integer $\lambda \in E$ a Pisot number if $\lambda > 1$ and for all monomorphisms $\sigma_i : E \to \mathbb{C}$ which are not the inclusion, it holds that $|\sigma_i(\lambda)| < 1$. An unit Pisot number is then an algebraic unit which is also a Pisot number. Note that unit Pisot number only exists in number fields E which are real, since otherwise the monomorphism $\sigma : E \to E$ given by complex conjugation satisfies $|\sigma(\lambda)| = |\lambda|$.

The last property for hyperbolic algebraic units we study is the full rank condition. Let λ be an algebraic unit with E-conjugates $\lambda_1 = \lambda, \dots, \lambda_n$, then λ satisfies the full rank condition if for all $d_1, \dots, d_n$ integers with

$$\prod_{j=1}^{n}\lambda_j^{d_j} = \pm 1,$$

it must hold that $d_1 = d_2 = \ldots = d_n$. The easiest examples of hyperbolic units satisfying the full rank condition are unit Pisot numbers.

Lemma 4.10. *Every Pisot number satisfies the full rank condition.*

Proof. Take a finite degree field extension F of E such that F is Galois over $\mathbb{Q}$. Assume that

$$\prod_{j=1}^{n}\lambda_j^{d_j} = \pm 1, \tag{4.2}$$

then we can assume that $d_j \geq 0$ for every j by multiplying equation (4.2) with the relation $\prod_{j=1}^{n}\lambda_j = \pm 1$.

Take $1 \leq k \leq n$ such that d_k is the maximal power occurring in equation (4.2) and fix $\sigma \in \mathrm{Gal}(F, \mathbb{Q})$ such that $\sigma(\lambda_k) = \lambda$. By applying σ to equation 4.2, we get a relation of the same form but with d_1 the maximal power occurring in the equation.

Since d_1 is the maximal element, we have that

$$1 = |\prod_{j=1}^{n}\lambda_j^{d_j}| \geq |\prod_{j=1}^{n}\lambda_j^{d_1}|$$

where equality holds if and only if $d_j = d_1$ since $|\lambda_j| < 1$. Because $|\prod_{j=1}^{n} \lambda_j^{d_1}| = 1$, the statement thus follows. $\square$

This easy lemma is also proved in [89, Proposition 3.6.].

To construct algebraic units which are c-hyperbolic or unit Pisot numbers we use two different techniques. Essentially, both methods start from the proof for Dirichlet Units theorem and then translates the additional property in an algebraic or topological condition for the lattice points. The first method translates the additional properties to linear equations in the vector space V. The topological method uses that the $\mathbb{Q}$-span of the lattice is dense in V and tries to describe the additional properties as an open subset of V.

Algebraic method

For the algebraic proof, we distinguish two cases, depending on the fact if the field E is totally imaginary or not. These proofs are analogous to the one in [39] which only treated the case of totally real Galois extensions over $\mathbb{Q}$.

We start with the following definition coming from geometric group theory which is used to compare the growth function of finitely generated groups. For more details about these growth functions, we refer to Chapter 11.

Definition 4.11. Let $\beta_1, \beta_2 : \mathbb{N} \to \mathbb{N}$ be two functions, then we say that β_2 **dominates** β_1, denoted as $\beta_1 \prec \beta_2$, if there exists $c > 0,\ d > 1$ such that

$$\beta_1(n) \leq c\beta_2(dn) + c$$

for every $n \in \mathbb{N}$. We say that β_1 and β_2 are **quasi-equivalent** and denote this as $\beta_1 \sim \beta_2$ if both $\beta_1 \prec \beta_2$ and $\beta_2 \prec \beta_1$.

By using this equivalence relation, the following lemma is immediate.

Lemma 4.12. *Let $L \subseteq \mathbb{R}^n$ be a uniform lattice and $V_1, \ldots, V_k$ a finite number of proper subspaces of $\mathbb{R}^n$. Then there always exists $x \in L$ such that*

$$x \notin \bigcup_{i=1}^{k} V_i.$$

Proof. The number of lattice points in a ball of radius r in $\mathbb{R}^n$ is quasi-equivalent to a polynomial of degree n. Similarly, the number of lattice points in a ball of radius r of a proper subspace is quasi-equivalent to a polynomial of degree strictly smaller than n. So by taking r big enough, there exists a lattice point in a ball of radius r which is not contained any of the proper subspaces, which implies the lemma. □

Proposition 4.13. *Let E be a number field of degree n which is not totally imaginary. Then there exists a c-hyperbolic $\mu \in U_E$ for all $c \leq n-1$.*

Proof. Of course it suffices to consider the case where $c = n-1$. Write $n = s+2t$ as before with $s \neq 0$ by the conditions of the theorem and consider the map $l : U_E \to \mathbb{R}^{s+t}$ as before. The image $l(U_E)$ is a uniform lattice in the vector space

$$V = \left\{ (x_1, \ldots, x_{s+t}) \in \mathbb{R}^{s+t} \;\middle|\; \sum_{i=1}^{s} x_i + \sum_{j=s+1}^{t} 2x_j = 0 \right\}$$

of dimension $s+t-1$ We claim that there exists an element $x \in l(U_E)$ for which

$$\forall k \in \{1, \ldots, c\},\ \forall i_1, \ldots, i_k \in \{1, \ldots, s+t\} : \ \sum_{j=1}^{k} x_{i_j} \neq 0.$$

Any element $\mu \in U_E$ with $l(\mu) = x$ is then obviously a c-hyperbolic element.

It's easy to see that each of the equations $\sum_{j=1}^{k} x_{i_j} = 0$ is linearly independent of the equation determining V. For this to be true, we indeed use the assumption that $c \leq n-1 = s+2t-1$ and $s > 0$. So the set of solutions of $\sum_{j=1}^{k} x_{i_j} = 0$ in V determines a subspace of V of dimension $s+t-2$. By Lemma 4.12 this implies that there exists $x \in l(U_E)$ such that x does not satisfy the equations above. □

Proposition 4.14. *Let E be a number field of degree n which is totally imaginary. Then there exists a c-hyperbolic $\mu \in U_K$ for all $c \leq \frac{n}{2} - 1$.*

The proof is completely the same as in the previous case and thus is left for the reader. As stated above, the bounds given in both propositions are optimal.

Topological method

The proofs of this part work for general real number fields, although they were only stated for real Galois extension in the original paper [42].

From Dirichlet's Unit Theorem we know that the map

$$l : U_E \to \mathbb{R}^n$$
$$\lambda \mapsto (\log|\sigma_1(\lambda)|, \ldots, \log|\sigma_s(\lambda)|, \log|\sigma_{s+1}(\lambda)|, \ldots, \log|\sigma_{s+t}(\lambda)|)$$

maps U_E onto a uniform lattice of the subspace $V \subseteq \mathbb{R}^n$ as in equation (4.1). By renumbering the monomorphisms, we can assume that σ_1 is the identity map. The unit Pisot numbers are mapped to the open subset $O \subseteq V$ given by the equations $x_1 > 0$ and $x_i < 0$ for all $i \geq 2$. So for the existence of unit Pisot numbers, one has to show that $O \cap l(U_E) \neq \varnothing$. The following lemma asserts that this is indeed the case.

Lemma 4.15. *Let $L \subseteq \mathbb{R}^n$ be a uniform lattice and $O \subseteq \mathbb{R}^n$ a nonempty open subset such that for all $v_1, v_2 \in O$ also $v_1 + v_2 \in O$. Then $O \cap L \neq \varnothing$.*

Proof. Since O is open and $L \otimes \mathbb{Q}$ is dense, there exists $x \in O \cap L \otimes \mathbb{Q}$. By taking $nx = x + \ldots + x$ for some $n \in \mathbb{N}_0$, we find $x \in L \cap O$. □

So this lemma implies that there exist unit Pisot numbers in every real number field $E \neq \mathbb{Q}$.

Theorem 4.16. *Let $E \subseteq \mathbb{R}$ be a real number field, then there exists a unit Pisot number in E.*

The lemma also implies that every open nonempty subset of V which is invariant under addition gives rise to possible algebraic units. For example, there also exists unit Pisot numbers with an extra condition on them:

Proposition 4.17. *Let E be a real number field and fix some monomorphism $\sigma : E \to \mathbb{C}$ which is not the inclusion map. Let $c \geq 1$ be any integer, then there always exists an unit Pisot number $\lambda \in E$ such that $|\sigma(\lambda^{c+1})\lambda^c| < 1$.*

Proof. Assume that σ_2 of the map l given above is equal to σ. Let $O \subseteq V$ be the open nonempty subset of V given by the inequalities $x_1 > 0, x_i < 0$ for all $i \geq 2$ and $cx_1 + (c+1)x_2 < 0$, then there exists $x \in O \cap l(U_E)$ because of the previous lemma. Any element of the preimage of x will satisfy the conditions of the proposition. □

If λ is an algebraic integer as in Proposition 4.17, then it also satisfies $|\sigma(\lambda^{k+1})\lambda^k| < 1$ for all $0 \leq k \leq c$, since

$$|\sigma(\lambda^{k+1})\lambda^k||\sigma(\lambda^{c-k})\lambda^{c-k}| = |\sigma(\lambda^{c+1})\lambda^c|$$

and $|\sigma(\lambda^{c-k})\lambda^{c-k}| \geq 1$ because λ is a unit Pisot number. The case $c = 1$ will be crucial for the construction of Anosov automorphisms of minimal signature, see Chapter 9 and therefore we state it separately here.

Corollary 4.18. Let E be a real number field and fix some $\sigma : E \to \mathbb{C}$ with σ not the inclusion map. Then there always exists an unit Pisot number $\lambda \in E$ such that $|\sigma(\lambda^2)\lambda| < 1$.

In Chapter 9 we only use these results for real Galois extension and the monomorphism σ is then given by a non-trivial element in the Galois group $\mathrm{Gal}(E, \mathbb{Q})$.

4.3 Existence of Galois extensions

In the previous section we studied the existence of c-hyperbolic units in number fields. Proposition 4.13 and 4.14 ensured the existence of such algebraic units when the degree of the field extension is sufficiently high. Therefore it is sometimes necessary to find (Galois) extensions of a given degree to guarantee the existence of such units for a given value of c.

The results of this chapter give an easy proof of a partial answer to the inverse Galois problem for cyclic groups.

Inverse Galois problem. Let G be a finite group and E a field, does there exists a Galois extension $F \supseteq E$ such that $\mathrm{Gal}(F, E) \simeq G$?

The case where $E = \mathbb{Q}$ is the most studied and there are many partial results in this case. The general problem is still widely open in algebraic number theory. A detailed introduction can be found in [103]. In this section we show that for every cyclic group $\mathbb{Z}_k$, the inverse Galois problem has a positive answer

Theorem 4.19. *Let E be a number field and $\mathbb{Z}_k$ a finite cyclic group. Then there exists a Galois extension $E \subseteq F$ such that $\mathrm{Gal}(F, E) \simeq \mathbb{Z}_k$. If E is real, then we can choose the field extension F to be real.*

For the field $\mathbb{Q}$, the proof is immediate by considering cyclotomic fields.

Example 4.20. Consider a primitive n-th root of unity ξ_n and the cyclotomic field $F = \mathbb{Q}(\xi_n)$. The minimal polynomial of ξ_n is equal to

$$p(X) = \sum_{i=0}^{\phi(n)} X^i$$

where ϕ is the Euler's totient function which is defined the fact that $\phi(n)$ is the order of the unit group $\mathbb{Z}_n^\times = U(n)$. We identify the group $U(n)$ as a subset of $\{1, \ldots, n-1\}$. The roots of the polynomial $p(X)$ are exactly the primitive n-th roots of unity, so they are all of the form ξ_n^i for $i \in U(n)$. This implies that F is a Galois extension of $\mathbb{Q}$ since it is the splitting field of the polynomial $p(X)$.

For every $i \in U(n)$, there is an automorphism $\sigma_i : F \to F$ given by $\sigma_i(\xi_n) = \xi_n^i$ on the generator ξ_n of F. The map $\psi : U(n) \to \mathrm{Gal}(F, \mathbb{Q})$ defined by

$$\psi(i) = \sigma_i$$

forms an isomorphism between $U(n)$ and $\mathrm{Gal}(F, \mathbb{Q})$.

Every finite cyclic subgroup $\mathbb{Z}_{2k}$ is isomorphic to a quotient of the group $U(4k^2)$. From the Galois correspondence it follows that there exists a Galois extension $K \supseteq \mathbb{Q}$ such that $\mathrm{Gal}(K, \mathbb{Q}) \simeq \mathbb{Z}_{2k}$. Let $\sigma : K \to K$ be the unique automorphism of order 2 in $\mathrm{Gal}(K, \mathbb{Q})$ and K' the field fixed by this element, so $K' = K^{\langle \sigma \rangle}$. From the Galois correspondence it follows that $\mathrm{Gal}(K', \mathbb{Q}) \simeq \mathbb{Z}_k$.

If K is real, then also K' is real and thus Theorem 4.19 holds. If K is not real, then σ is equal to complex conjugation on K and thus $K' = K \cap \mathbb{R}$ is real.

By using Example 4.20 we find the general version of Theorem 4.19.

Proof of Theorem 4.19. Let n be the degree of the field extension $\mathbb{Q} \subseteq E$. From Example 4.20 we know that there exists a field extension $\mathbb{Q} \subseteq K$ for every given degree such that K is Galois over $\mathbb{Q}$ and with $\mathrm{Gal}(K, \mathbb{Q})$ a cyclic group.

Let K be such a field extension with $[K : \mathbb{Q}] = mn$. Look at the field $F = EK$, the smallest field which contains both E and K. Since K is a splitting field over $\mathbb{Q}$ of a polynomial $f \in \mathbb{Q}[X]$, F is also a splitting field of the same polynomial f over the field E. Thus F is a Galois extension of E. Moreover, since

$$mn = [K : \mathbb{Q}] \mid [F : \mathbb{Q}] = [F : E][E : \mathbb{Q}]$$

and $[E : \mathbb{Q}] = n$, it holds that $m \mid [F : E]$.

We now claim that $\mathrm{Gal}(F, E)$ is cyclic. Consider the group morphism

$$\pi : \mathrm{Gal}(F, E) \to \mathrm{Gal}(K, \mathbb{Q}) : \sigma \mapsto \sigma|_K.$$

The group morphism π is injective. Indeed, assume that $\pi(\sigma) = \mathbb{1}_K$ for $\sigma \in \mathrm{Gal}(F, E)$ and take $x = yz \in F = EK$ with $y \in E, z \in K$, then

$$\sigma(x) = \sigma(yz) = \sigma(y)\sigma(z) = yz = x$$

because σ is the identity both on E and K. Since $\mathrm{Gal}(K, \mathbb{Q})$ is cyclic, therefore also $\mathrm{Gal}(F, E)$ must be cyclic.

The group $\mathrm{Gal}(F, E)$ is a finite cyclic group and $m \mid |\mathrm{Gal}(F, E)|$, therefore we can always find a (normal) subgroup $H \leq \mathrm{Gal}(F, E)$ of index m. By the fundamental theorem of Galois theory, there exists a subfield F_0 of F, Galois over E, such that $[F_0 : E] = m$ and thus the field F_0 satisfies the conditions of the theorem. If E is real, then since K is also real, we have that F and so also L_0 is realized as a subfield of $\mathbb{R}$. $\square$

This weaker version of Theorem 4.19 is the only result we need in this dissertation.

Corollary 4.21. Let E be a number field and $m > 0$ a natural number. Then there exists a Galois extension $E \subseteq F$ such that $[F : E] = m$. If E is real, then there exists a field extension F which is real.

Chapter 5

Subgroups of the matrix group $\mathrm{GL}(n, \mathbb{C})$

This chapter discusses various topics related to subgroups of the matrix group $\mathrm{GL}(n, \mathbb{C})$. The first section deals with representations of finite groups, which are group morphisms from these groups to the general linear group $\mathrm{GL}(n, \mathbb{C})$. The image of a representation forms a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. Next we discuss some canonical forms of matrices in $\mathrm{GL}(n, \mathbb{C})$ which are a useful tool for determining which matrices are conjugate. Since every matrix of $\mathrm{GL}(n, \mathbb{C})$ generates a cyclic subgroup, this corresponds to studying which cyclic subgroups of $\mathrm{GL}(n, \mathbb{C})$ are conjugate. Finally we introduce linear algebraic K-groups over a field $K \subseteq \mathbb{C}$ which are subgroups of $\mathrm{GL}(n, \mathbb{C})$ given by the zero set of polynomials over K. An important ingredient is the multiplicative Jordan decomposition which decomposes every element of $\mathrm{GL}(n, \mathbb{C})$ into its unipotent and semisimple part. In short, this chapter discusses three types of subgroups of $\mathrm{GL}(n, \mathbb{C})$, namely finite, cyclic and K-closed subgroups.

5.1 Representation theory for finite groups

During this section, H is always a finite group and a vector space over the field K is denoted as V^K. The group of invertible linear transformations of V^K is written as $\mathrm{GL}(V^K)$. Most of the time we restrict our attention to the vector space K^n and then we write the group of automorphisms as $\mathrm{GL}(n, K)$. By fixing a basis for V^K, every representation is of this form, so there is no loss in generality by restricting to $\mathrm{GL}(n, K)$.

A representation of the group H is a group morphism

$$\rho : H \to \mathrm{GL}(V^K)$$

from H to the group $\mathrm{GL}(V^K)$. Let $\rho_1 : H \to \mathrm{GL}(V_1^K)$ and $\rho_2 : H \to \mathrm{GL}(V_2^K)$ be two representations of the group H, then a H-morphism between ρ_1 and ρ_2 is a linear map $\varphi : V_1^K \to V_2^K$ such that

$$\varphi \circ \rho_1(h) = \rho_2(h) \circ \varphi$$

for every $h \in H$. If the map φ is an isomorphism, then we call the representation ρ_1 and ρ_2 equivalent. In the case $V^K = K^n$, two representations $\rho_1 : H \to \mathrm{GL}(n, K)$ and $\rho_2 : H \to \mathrm{GL}(n, K)$ are equivalent if and only if there exists a $P \in \mathrm{GL}(n, K)$ such that

$$\rho_1(h) = P\rho_2(h)P^{-1}$$

for every $h \in H$.

Let $\rho : H \to \mathrm{GL}(n, K)$ be a representation, then a subrepresentation is a subspace $V^K \subseteq K^n$ such that $\rho(h)(V^K) = V^K$ for every h in H. For every representation, there are at least two trivial subrepresentations given by the subspaces 0 and K^n. A representation is called irreducible if it has no other subrepresentations than the trivial ones. The following classical result describes the H-morphisms between irreducible representations.

Lemma 5.1 (Schur's Lemma). *Let $\rho_1 : H \to \mathrm{GL}(V_1^K)$ and $\rho_2 : H \to \mathrm{GL}(V_2^K)$ be irreducible representations of the group H, then every non-zero H-morphism between ρ_1 and ρ_2 is an isomorphism.*

The irreducible representations of H form the building blocks of general representations $\rho : H \to \mathrm{GL}(n, K)$. The following result shows that every representation which is not irreducible can be decomposed in smaller representations.

Theorem 5.2 (Maschke's Theorem). *Let $\rho : H \to \mathrm{GL}(n, K)$ be a representation and $V^K \subseteq K^n$ a subrepresentation. Then there always exists a complementary subspace $W^K \subseteq K^n$, i.e. with $V^K \oplus W^K = K^n$, such that $\rho(h)(W^K) = W^K$ for every $h \in H$.*

There is a more general version of Maschke's Theorem for fields of arbitrary characteristic, see [68], but we only need the version for characteristic 0 in this thesis. By Maschke's Theorem, every representation can be decomposed into irreducible subrepresentations and this decomposition is unique up to permutation of the representations. Sometimes we will talk about K-equivalent and K-irreducible representations to emphasize we are working over the field K.

Let $\rho : H \to \mathrm{GL}(n, K)$ be a representation and $K \subseteq L$ a field extension of K. By considering the group $\mathrm{GL}(n, K)$ as a subgroup of $\mathrm{GL}(n, L)$, there is a natural representation $\rho : H \to \mathrm{GL}(n, L)$ which we denote as ρ^L. Equivalently, ρ^L is the representation we get by extending the scalars of the vector space K^n.

Every representation contains a lot of redundant information, therefore we introduce characters of representations. The character corresponding to a representation $\rho : H \to \mathrm{GL}(n, K)$ is the class function $\chi : H \to K$ given by

$$\chi(h) = \mathrm{Tr}(\rho(h)).$$

A character is called irreducible if it is the character of an irreducible representation. If two representations ρ_1 and ρ_2 have the same character, the representations ρ_1 and ρ_2 are equivalent. If K is algebraically closed, this is a standard result, for arbitrary fields this is a consequence of [68, Corollary 9.7.].

Denote by $\mathrm{Irr}(H)$ the set of $\mathbb{C}$-irreducible characters of H. Let K be a field and χ a character (not necessarily irreducible), then we say that χ is afforded by a K-representation if there exists a representation $\rho : H \to \mathrm{GL}(n, K)$ such that the character of ρ equals χ. If χ is an irreducible character, we write $\mathbb{Q}(\chi)$ for the smallest subfield of $\mathbb{C}$ that contains $\chi(h)$ for all $h \in H$. The field $K = \mathbb{Q}(\chi)$ is Galois over $\mathbb{Q}$ and for every $\sigma \in \mathrm{Gal}(K, \mathbb{Q})$, we can define the map

$$\begin{aligned} \chi^\sigma : H &\to K \\ h &\mapsto \sigma(\chi(h)), \end{aligned}$$

which we call a Galois conjugate of χ. Every Galois conjugate of χ is also an irreducible character of H because of [68, Lemma 9.16].

Let ρ be a $\mathbb{Q}$-irreducible representation of a finite group H. From [68, Theorem 9.21] we know that each $\mathbb{C}$-irreducible component of ρ occurs with the same multiplicity m. If $\chi \in \mathrm{Irr}(H)$ is the character of one of those components, then the set of characters of all $\mathbb{C}$-irreducible components of ρ is given by the Galois conjugates of χ. So if we have $\{\sigma_1, \ldots, \sigma_n\} = \mathrm{Gal}(\mathbb{Q}(\chi), \mathbb{Q})$, we can write the character χ_ρ of ρ as

$$\chi_\rho = m\chi^{\sigma_1} \oplus \ldots \oplus m\chi^{\sigma_n}.$$

The fixed multiplicity m is called the Schur index of χ over $\mathbb{Q}$ and is written as $m_{\mathbb{Q}}(\chi)$ (see [68, pages 160–161]). The number $m_{\mathbb{Q}}(\chi)$ is also the smallest integer m such that $m\chi$ is afforded by a $\mathbb{Q}(\chi)$–representation. Moreover, it is also given by the minimal degree of a field extension $\mathbb{Q}(\chi) \subseteq F$ such that χ is afforded by an F-representation ([68, Theorem 10.17]). Such a field F is called a minimal splitting field for the character χ. Note that the number of $\mathbb{C}$-irreducible components of ρ is equal to $m \cdot n = [F : \mathbb{Q}(\chi)] \cdot [\mathbb{Q}(\chi) : \mathbb{Q}] = [F : \mathbb{Q}]$.

5.2 Canonical forms of matrices

Let K be a field of characteristic 0 and $\mathrm{End}(K^n) = K^{n\times n}$ the set of $n \times n$ matrices over K. Two matrices $A, B \in K^{n\times n}$ are called similar over K if there exists an invertible matrix $P \in \mathrm{GL}(n, K)$ such that

$$PAP^{-1} = B.$$

One result of this section is that similarity is invariant under field extension. So if $A, B \in K^{n\times n}$ are similar over L with $L \supseteq K$ a field extension, then the matrices A and B are also similar over K.

A standard problem in matrix theory is to determine whether two matrices are similar. One technique for solving this problem is to consider canonical forms associated to the matrices A and B. These canonical forms can be computed algorithmically and two matrices A and B are similar if and only if they have the same canonical form. We give a few examples of these canonical forms. For more details and proofs we refer to [66].

Jordan canonical form

If K is algebraically closed and $A \in K^{n\times n}$ is semisimple, then there exists an invertible matrix $P \in \mathrm{GL}(n, K)$ such that PAP^{-1} is given by diagonal matrix. If A is not semisimple, e.g. the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then this is not possible. The Jordan canonical form combines these insights and shows that any matrix over an algebraically closed field is built from these two examples.

A vector $v \in K^n$ is called an eigenvector for the eigenvalue $\lambda \in K$ if $A(v) = \lambda v$ or equivalently in the case that

$$(A - \lambda \mathbb{1}_n)(v) = 0.$$

A vector $v \in K^n$ is called a generalized eigenvector for the eigenvalue λ if there exists some $k > 0$ such that

$$(A - \lambda \mathbb{1}_n)^k (v) = 0.$$

This means that v is eventually mapped to an eigenvector under the linear map $(A - \lambda \mathbb{1}_n)$. The set of all generalized eigenvectors for an eigenvalue λ is a subspace of K^n, which we call the generalized eigenspace for the eigenvalue λ and denote this subspace as V_λ. The vector space V_λ is non-trivial if and only if λ is an eigenvalue of A.

For a semisimple matrix, the vector space K^n is equal to the direct sum of its eigenspaces. For general matrices, there is the following extension.

Theorem 5.3. *Let $A \in K^{n\times n}$ where K is algebraically closed and V_λ the generalized eigenspace for eigenvalue λ. Then*

$$K^n = \bigoplus_{\lambda \in K} V_\lambda.$$

The Jordan canonical form is then the matrix we get by taking a suitable basis for the generalized eigenspaces.

Theorem 5.4. *Let $A \in K^{n\times n}$ over an algebraically closed field K, then A is similar over K to a matrix of the form*

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}$$

where every matrix J_i is of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}.$$

Moreover this matrix is unique op to permutation of the matrices J_i.

The matrices J_i are called the Jordan blocks of the matrix A. The uniqueness of the canonical form is what makes it possible to check whether two matrices are similar over K.

For some fields K which are not algebraically closed, we have a similar decomposition. For example, if $K = \mathbb{R}$, we have the following real Jordan canonical form.

Theorem 5.5. *Let $A \in \mathbb{R}^{n\times n}$, then A is similar over $\mathbb{R}$ to a matrix of the form*

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}$$

where every matrix J_i is a Jordan block for a real eigenvalue or of the form

$$J_i = \begin{pmatrix} C_i & \mathbb{1}_2 & & \\ & C_i & \ddots & \\ & & \ddots & \mathbb{1}_2 \\ & & & C_i \end{pmatrix}$$

where $C_i = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$ has eigenvalues $a_i \pm ib_i$. Moreover this matrix is unique op to permutation of the matrices J_i.

For most fields K it is impossible to find such a Jordan canonical forms though, so therefore we consider also other canonical forms. We are particularly interested in the case $K = \mathbb{Q}$, which we discuss in the following paragraph.

Rational canonical form

This canonical form exists for every field K, although we only define it here for the rationals $K = \mathbb{Q}$. Every matrix $A \in \mathbb{Q}^{n\times n}$ makes $\mathbb{Q}^n$ into a $\mathbb{Q}[X]$ module by letting X act as A on $\mathbb{Q}^n$, i.e. for every polynomial

$$p(X) = q_0 + q_1 X + \ldots + q_k X^k \in \mathbb{Q}[X],$$

the action is given by

$$^{p(X)}v = \sum_{i=0}^{k} q_i A^i(v).$$

Since $\mathbb{Q}[X]$ is a principal ideal domain, we can use the structure theorem for finitely generated modules over a principal ideal domain.

Theorem 5.6. *Let M be a finitely generated module over a principal ideal domain R. Then there is a unique decreasing sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that M isomorphic to the sum of cyclic modules.*

$$M \simeq \bigoplus_{i=1}^{n} R/(d_i) = R/(d_1) \oplus R/(d_2) \oplus \cdots \oplus R/(d_n).$$

Recall that the companion matrix $L(f)$ of a monic polynomial

$$f(X) = \sum_{i=0}^{k} a_i X^i \in K[X]$$

is the matrix of the form

$$L(f) = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{k-1} \end{pmatrix}.$$

The characteristic polynomial of $L(f)$ is equal to $f(X)$.

By applying Theorem 5.6 to the $\mathbb{Q}[X]$ module defined from a matrix $A \in \mathbb{Q}^{n\times n}$ and by choosing an appropriate basis for the cyclic modules, we get the following canonical form.

Theorem 5.7. *Every matrix $A \in \mathbb{Q}^{n\times n}$ is similar over $\mathbb{Q}$ to a matrix of the form*

$$\begin{pmatrix} L(f_1) & 0 & \dots & 0 \\ 0 & L(f_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L(f_k) \end{pmatrix}$$

for monic polynomials $f_i \in \mathbb{Q}[X]$ such that $f_i \mid f_{i+1}$ in $\mathbb{Q}[X]$ where $L(f_i)$ is the companion matrix of f_i. These polynomials f_i are uniquely determined by the matrix A and are called the invariant factors of A.

We define the matrix of Theorem 5.7 as the rational canonical form of A. From the theorem, it follows that f_k is the minimal polynomial and $f_1 \dots f_k$ is the characteristic polynomial of A. In particular, all polynomials f_i divide the characteristic polynomial of A.

There is an algorithm to compute the invariant factors of a matrix A. Therefore Theorem 5.7 can be used to check whether two matrices are similar over $\mathbb{Q}$. But it also has the following consequence for studying the similarity between matrices.

Corollary 5.8. Let $A, B \in \mathbb{Q}^{n\times n}$ be rational matrices and assume that A and B are similar over $\mathbb{C}$. Then A and B are also similar over $\mathbb{Q}$.

Proof. Since A and B are similar over $\mathbb{C}$, their invariant factors are equal. Theorem 5.7 then implies that they have the same rational canonical form and thus are similar over $\mathbb{Q}$. □

This proof works for general fields K of characteristic 0. So instead of checking whether two matrices A and B are similar over K, we can check this over the algebraic closure $\bar{K}$ of K and hence we can use the Jordan normal form. In particular, the Jordan normal form of a matrix $A \in \mathrm{GL}(n,K)$ exists for every field K which contains every eigenvalue of A.

The following consequence of the rational canonical form is crucial for this thesis.

Proposition 5.9. *Let $A \in \mathrm{GL}(n,\mathbb{Q})$ be a matrix with characteristic polynomial $p(X) \in \mathbb{Z}[X]$, then there exists an invertible matrix $P \in \mathrm{GL}(n,\mathbb{Q})$ such that PAP^{-1} has entries in $\mathbb{Z}$.*

Proof. First we show that every monic polynomial $q(X) \in \mathbb{Q}[X]$ which divides p is also an element of $\mathbb{Z}[X]$. Note that every root of p is an algebraic integer by definition of algebraic integers. First assume that q is irreducible, then $q(X)$ is the minimal polynomial of an algebraic integer and thus lies in the ring $\mathbb{Z}[X]$. For the general case, write q as a product of its $\mathbb{Q}$-irreducible components which are elements of $\mathbb{Z}[X]$ to conclude the claim.

So the invariant factors of A are polynomials in $\mathbb{Z}[X]$. By Theorem 5.7 this implies that A is similar to a matrix with entries in $\mathbb{Z}$, so this finishes the proof. □

Let A be as in Proposition 5.9 and assume that $\det(A) = \pm 1$, then $PAP^{-1} \in \mathrm{GL}(n,\mathbb{Z})$. So the integer-like matrices of $\mathrm{GL}(n,\mathbb{Q})$ are exactly the matrices conjugate to an element of $\mathrm{GL}(n,\mathbb{Z})$.

Primary rational canonical form

By using the primary decomposition theorem for finitely generated modules over principal ideal domains, we can get a slightly different canonical form for rational matrices.

Theorem 5.10. *Every finitely generated module M over a principal ideal domain R is isomorphic to one of the form*

$$M \simeq \bigoplus_i R/(q_i)$$

where the ideals $(q_i) \neq R$ are primary ideals. The q_i are unique up to multiplication by units.

A primary ideal of R is an ideal I such that if $xy \in I$, then $x \in I$ or $y^n \in I$. The primary ideals in $K[X]$ are generated by the powers of irreducible polynomials in $K[X]$. In particular, we get the following primary rational canonical form matrices over $\mathbb{Q}$.

Theorem 5.11. *Every matrix $A \in \mathbb{Q}^{n\times n}$ is similar over $\mathbb{Q}$ to a matrix of the form*

$$\begin{pmatrix} L(p_1^{n_1}) & 0 & \dots & 0 \\ 0 & L(p_2^{n_2}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L(p_k^{n_k}) \end{pmatrix}$$

for irreducible polynomials $p_i \in \mathbb{Q}[X]$ where $L(p_i^{n_i})$ is the companion matrix of $p_i^{n_i}$. These polynomials p_i and powers n_i are uniquely determined by the matrix A up to permutation.

By rewriting the matrices $L(p_i^{n_i})$ we could write these matrices in a form similar to the Jordan canonical form, which is why the primary rational canonical form is sometimes called the Jordan rational canonical form. For more details, we refer to the book [66].

The importance of rational canonical forms in this thesis are twofold. On the one hand Theorem 5.7 implies that every matrix with characteristic polynomial in $\mathbb{Z}[X]$ is similar over $\mathbb{Q}$ to a matrix of $\mathrm{GL}(n, \mathbb{Z})$, which we will prove in Proposition 5.9. This will be useful to construct group morphisms from automorphisms of a Lie algebra with characteristic polynomial in $\mathbb{Z}[X]$. The second reason is to construct a matrix with only eigenvalues in $\mathbb{Q}$ starting from a matrix $A \in \mathrm{GL}(n, \mathbb{Q})$. This is crucial for constructing positive gradings starting from an expanding map, see Chapter 8.

5.3 Linear algebraic groups

In this section, we recall some basic properties about linear algebraic K-groups. A more detailed discussion can be found e.g. in [10, 65, 67]. We restrict our attention to subfield $K \subseteq \mathbb{C}$ of the complex numbers, since only these fields are important when studying infra-nilmanifolds. An important ingredient of this thesis is the multiplicative Jordan decomposition, which also exists in linear algebraic groups. One of the consequences is that we can always assume that elements in a linear algebraic group defined by eigenvalues are semisimple.

Linear algebraic groups

Consider a vector space V^K over the field K. A polynomial function $V^K \to K$ is a map which is given by a polynomial in the coordinates function for some basis of the vector space V^K (and thus for every basis of V^K). The ring of all polynomial function $V^K \to K$ is denoted as $K[V^K]$ and $K[V^K] \simeq K[x_1, \ldots x_n]$ with n the dimension of V^K. A polynomial function of $\mathrm{GL}(V^K)$ is a map $\mathrm{GL}(V^K) \to K$ which is the restriction of a polynomial function of the vector space $\mathrm{End}(K^n) \supseteq \mathrm{GL}(V^K)$.

A K-closed subset of $\mathrm{GL}(n, \mathbb{C})$ is the zero set of a finite number of polynomial functions with coefficients in K. A linear algebraic K-group G is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ which is also K-closed. We denote by $G(K) = G \cap \mathrm{GL}(n, K)$ the subgroup of K-rational points in G. The set $G(K)$ forms a Zariski-dense subset of G, see [67, Section 34.4] and sometimes we will identify a $\mathbb{Q}$-group with its subgroup of $\mathbb{Q}$-rational points.

The connected component of the identity element in G is denoted as G^0 and this subspace is a normal subgroup of finite index in G which also forms a linear algebraic K-group. A group morphism between two linear algebraic K-groups is said to be defined over K (or is a K-morphism) if the coordinate functions are given by polynomials over the field K. The multiplicative group $\mathbb{C}^*$ of the field $\mathbb{C}$ is a linear algebraic K-group for every subfield $K \subseteq \mathbb{C}$. A character of G is a group morphism $G \to \mathbb{C}^*$ defined over $\mathbb{C}$.

A K-torus is a linear algebraic K-group which is isomorphic to a closed subgroup of diagonal matrices $D(n, \mathbb{C})$. If the isomorphism is defined over K, then we call the torus K-split. If a K-torus has no non-trivial characters defined over K, then we call the torus anisotropic. All maximal tori of a connected linear algebraic K-group are conjugate and there always exists a maximal torus which is defined over K.

Example 5.12. Consider the group

$$G = \mathrm{SL}(n, \mathbb{C}) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \det(A) = 1\}$$

which is a connected linear algebraic group over every field $K \subseteq \mathbb{C}$, since the determinant $\det(A)$ of a matrix can be expressed as a polynomial over $\mathbb{Q}$.

One example of a maximal torus is the subgroup $T_1 \leq G$ given by

$$T_1 = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \,\middle|\, c \in \mathbb{C}^\times \right\},$$

which is defined over every field K. This is a K-split torus by construction. Another example is the torus

$$T_2 = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in G \,\middle|\, a, b, c \in \mathbb{C}, a = b + c, \right\},$$

which is the centralizer of the element $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ in G. The torus T_2 is anisotropic over $\mathbb{Q}$, but is K-split over $K = \mathbb{Q}(\sqrt{5})$ which is also the splitting field of the characteristic polynomial of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

The radical $\mathfrak{R}G$ of a linear algebraic K-group G is defined as the maximal connected and solvable normal subgroup of G. This normal subgroup always exists and is defined over K, see [10, Section 11.21]. The subgroup of $\mathfrak{R}G$ consisting of all its unipotent elements is called the unipotent radical and we denote this normal subgroup as $(\mathfrak{R}G)_u$. If the unipotent radical is trival, i.e. if $(\mathfrak{R}G)_u = \{e\}$, the group G is called reductive. A Levi subgroup is a (reductive) linear algebraic subgroup $L \subseteq G$ such that G is the semi-direct product of L and $(\mathfrak{R}G)_u$. From [65, Chapter VIII] we know that Levi subgroups always exist in characteristic 0 and that they are unique up to conjugation. Moreover, every reductive linear algebraic subgroup of G is contained in a Levi subgroup. If G is a connected reductive group, it can be decomposed as a product $G = Z \cdot \mathfrak{D}G$ where $Z = Z(G)^0$ is the identity component of the center of G and $\mathfrak{D}G = [G, G]$ is the commutator subgroup, see [10, Section 14.2]. The intersection $Z \cap \mathfrak{D}G$ is a finite subgroup of G.

Example 5.13. Let $E \subseteq \mathbb{C}$ be any field and $\mathfrak{n}^E$ a nilpotent Lie algebra over the field E. Consider the complexification $\mathfrak{n}^\mathbb{C}$ of the Lie algebra $\mathfrak{n}^E$. The automorphism group $G = \mathrm{Aut}(\mathfrak{n}^\mathbb{C})$ is a linear algebraic E-group and we have that $\mathrm{Aut}(\mathfrak{n}^E) = G(E)$.

Jordan decomposition

Let V^K be a finite dimensional vector space over K and denote by $\mathrm{End}(V^K)$ the set of all linear maps $V \to V$. Write $0 \in \mathrm{End}(V^K)$ for the trivial endomorphism on V^K which maps every element to 0.

Take $x \in \mathrm{End}(V^K)$, then we call a subspace $W \subseteq V$ invariant under x if $x(W) \subseteq W$. The endomorphism x is nilpotent if there exists some $k \in \mathbb{N}$ such that $x^k = 0$ or equivalently if x only has eigenvalues equal to 0. At the other extreme, we call x semisimple if for every subspace W invariant under x, there exists a complementary subspace W', i.e such that $W \oplus W' = V$, which is invariant under x. Equivalently, x is semisimple if its minimal polynomial has distinct roots or thus if x is diagonalizable over some field extension of K.

The additive Jordan decomposition states that every endomorphism x can be uniquely written as the sum of a nilpotent and a semisimple endomorphism.

Theorem 5.14 (Additive Jordan decomposition)**.** *Let $x \in \mathrm{End}(V^K)$, then there is a unique decomposition*

$$x = x_s + x_n$$

where $x_s, x_n \in \mathrm{End}(V^K)$ commute, x_s is semisimple and x_n is nilpotent. Moreover, x_s and x_n can be expressed as a polynomial over the algebraic closure of K evaluated at the original endomorphism x.

Proof. Note that 0 is the only endomorphism of V^K which is both nilpotent and semisimple, which already implies the uniqueness of the decomposition. If the field K contains all the eigenvalues of the endomorphism x, then the existence follows immediately from Theorem 5.4.

For general fields K, let L be a Galois extension which contains all the eigenvalues of x. By considering x as an element of $\mathrm{End}(V^L)$, we find unique $x_s, x_n \in \mathrm{End}(V^L)$ with x_s semisimple and x_n nilpotent such that $x = x_s + x_n$. For every $\sigma \in \mathrm{Gal}(L, K)$, it holds that

$$x = \sigma(x) = \sigma(x_s) + \sigma(x_u).$$

Because $\sigma(x_s)$ is semisimple and $\sigma(x_n)$ nilpotent, the uniqueness of the decomposition implies $\sigma(x_n) = x_n$ and $\sigma(x_s) = x_s$. Since this holds for every $\sigma \in \mathrm{Gal}(L, K)$ we conclude that $x_s, x_n \in \mathrm{End}(V^K)$.

The last statement about expressing x_s and x_n as a polynomial is left for the reader. $\square$

If $y \in \mathrm{End}(V^K)$ commutes with x, then y also commutes with x_s and x_n because of the last statement in the theorem.

An endomorphism x of V^K is called unipotent if $x - \mathbb{1}_{V^K}$ is nilpotent. Equivalently, x is unipotent if it only has eigenvalues equal to 1. In the remaining part of this thesis, the following consequence of the additive Jordan decomposition will be useful. This result is also called the multiplicative Jordan decomposition.

Theorem 5.15 (Multiplicative Jordan decomposition)**.** *Let $x \in \mathrm{GL}(V^K)$ be an automorphism of a finite dimensional vector space V^K over K. Then there exist unique $x_s, x_u \in \mathrm{GL}(V^K)$ with x_s semisimple and x_u unipotent such that $x = x_u x_s$ and $x_u x_s = x_s x_u$. These x_u and x_s satisfy the following properties.*

(1) *If $y \in \mathrm{End}(V^K)$ commutes with x, then y also commutes with the semisimple part x_s and the unipotent part x_u;*

(2) *If W is a subspace which is invariant under x, then W is also invariant under x_s and x_u;*

(3) *If $p \in K[\mathrm{GL}(V^K)]$ is a polynomial such that $p(x) = 0$, then $p(x_s) = 0 = p(x_u)$.*

Proof. If the decomposition exists, it is unique since $\mathbb{1}_V$ is the only element in $\mathrm{GL}(V^K)$ which is both unipotent and semisimple. For the existence, write $x = x_s + x_n$ as in Theorem 5.14. Since x, x_s, x_n commute, x_s has the same eigenvalues as x and thus is also invertible. Since x_s and x_n commute, the endomorphism $x_s^{-1}x_n$ is nilpotent. Take $x_u = x_s^{-1}x = 1 + x_s^{-1}x_n$ which is unipotent. It then holds that $x = x_s x_u$ and x_u and x_s obviously commute. The other properties then follow from Theorem 5.14 and the fact that x_s and x_n can be expressed as polynomials in x. □

The last statement of Theorem 5.15 implies that the multiplicative Jordan decomposition also exists in linear algebraic K-groups. Since x_s and x_u commute and x_u only has eigenvalues equal to 1, the eigenvalues of x_s are identical to the eigenvalues of x. In this thesis we are interested in certain elements of linear algebraic K-groups which are defined using their eigenvalues. Theorem 5.15 then implies that we can always assume that these elements are semisimple.

If $x \in \mathrm{GL}(V^K)$ is an element of finite order, then x is always semisimple. Indeed, write $x = x_s x_u$ and k the order of x, then

$$\mathbb{1}_{V^K} = x^k = x_s^k x_u^k$$

and thus $x_u^k = 0$. This implies that $x_u = 0$ or thus that $x = x_s$ is semisimple. In particular, every element in the image of a representation of a finite group is semisimple.

Part II

Expanding maps

In this second part of the thesis, we give a complete algebraic characterization of the infra-nilmanifolds admitting an expanding map. The main result, which we prove in Chapter 8, shows that the existence of such an expanding map depends only on the covering Lie group.

Main Theorem 1. *Let $\Gamma\backslash G$ be an infra-nilmanifold modeled on the Lie group G with corresponding Lie algebra $\mathfrak{g}$. Then the following are equivalent:*

(1) *The infra-nilmanifold $\Gamma\backslash G$ admits an expanding map;*

(2) *The Lie algebra $\mathfrak{g}$ has a positive grading;*

(3) *The Lie group G has an expanding automorphism.*

As an introduction, we give a sketch of the proof of Theorem 8.21 in the more restrictive case of nilmanifolds.

Let N be an $\mathcal{F}$-group and denote by $\mathrm{Endo}(N)$ the set of injective group morphisms of N. Every injective group morphism of N induces an automorphism of the Lie group G and thus there is a natural inclusion

$$\mathrm{Endo}(N) \subseteq \mathrm{Aut}(G).$$

Therefore every expanding nilmanifold endomorphism induces an expanding automorphism of the Lie group G and an expanding automorphism on the Lie algebra $\mathfrak{g}$ corresponding to G as well, which is exactly the implication from (1) to (3).

The set $\mathrm{Endo}(N)$ is a discrete subset of the group $\mathrm{Aut}(G)$ and thus most elements of $\mathrm{Aut}(G)$ are not induced by an injective group morphism of N. Hence the hard part of Main Theorem 1 is to construct an expanding map on $N\backslash G$ starting from an expanding automorphism of G. This is equivalent to constructing an expanding automorphism $\varphi : G \to G$ such that $\varphi(N) \leq N$.

Said differently, we want to cross the gap between $\mathrm{Endo}(N)$ and $\mathrm{Aut}(G)$ and construct an injective group morphism in $\mathrm{Endo}(N)$ starting from an automorphism in $\mathrm{Aut}(G)$. To solve this problem, we put the group $\mathrm{Aut}(N^{\mathbb{Q}})$ in between the set $\mathrm{Endo}(N)$ and the group $\mathrm{Aut}(G)$, so

$$\mathrm{Endo}(N) \subseteq \mathrm{Aut}(N^{\mathbb{Q}}) \leq \mathrm{Aut}(G). \tag{5.1}$$

The main idea of the proof is to first construct an expanding automorphism of $N^{\mathbb{Q}}$ and use this expanding automorphism to construct an injective group morphism in $\mathrm{Endo}(N)$. So the gap between the set $\mathrm{Endo}(N)$ and $\mathrm{Aut}(G)$ is split into two smaller gaps and each of these gaps can be crossed by different methods which we will explain in the following chapters.

The first step is to determine the relation between $\mathrm{Aut}(N^{\mathbb{Q}})$ and $\mathrm{Endo}(N)$ for a fixed full subgroup N of $N^{\mathbb{Q}}$. As we explain in Chapter 6 this corresponds to showing that the existence of an expanding map on a nilmanifold only depends on the commensurability class of the fundamental group. The methods of this chapter use the structure of $\mathcal{F}$-groups and more specific the existence of lattice groups as introduced in Section 2.5. In Chapter 7 we will use the same techniques to study periodic points of affine infra-nilmanifold endomorphisms.

The second step is then the connection between automorphisms of $N^{\mathbb{Q}}$ and G. If we consider the corresponding Lie algebras $\mathfrak{n}^{\mathbb{Q}}$ and $\mathfrak{g}$, this is equivalent to studying automorphisms of rational forms of real nilpotent Lie algebras. The group $\mathrm{Aut}(\mathfrak{g}) \simeq \mathrm{Aut}(G)$ forms a linear algebraic $\mathbb{Q}$-group with the group $\mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}}) \simeq \mathrm{Aut}(N^{\mathbb{Q}})$ as its subgroup of $\mathbb{Q}$-rational points. Therefore all techniques about linear algebraic groups are available to study these groups, as explained in Section 5.3. Chapter 8 discusses these results in detail.

The proof for general infra-nilmanifolds follows the same steps, but in this case the rational holonomy group $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$ plays an important role. This is comparable to the situation of Anosov diffeomorphisms in Theorem 3.36.

There is a similar algebraic characterization for the existence of non-trivial self-covers on infra-nilmanifolds.

Main Theorem 2. *Let $\Gamma \backslash G$ be an infra-nilmanifold modeled on a Lie group G with corresponding Lie algebra $\mathfrak{g}$. Then the following are equivalent:*

(1) *The infra-nilmanifold $\Gamma \backslash G$ admits a non-trivial self-cover;*

(2) *The group Γ is not cohopfian;*

(3) *The Lie algebra $\mathfrak{g}$ has a non-trivial non-negative grading;*

(4) *The Lie group G has a partially expanding automorphism.*

The techniques to prove this result are more or less the same as the ones for expanding maps.

At this point, we want to remark that parts of Main Theorem 1 and Main Theorem 2 were proved independently and by different methods by Y. Cornulier in [20]. This paper only considers nilmanifolds and studies a property called being dis-cohopfian, which is the translation of the existence of an expanding map to the level of groups. We will say more about this property and the relation with expanding maps in Chapter 11.

Chapter 6

Expanding maps and the rational holonomy representation

In Chapter 2 we introduced the rational holonomy representation of an almost-Bieberbach group. If Γ is an almost-Bieberbach group with holonomy group F and Fitting subgroup N, then this representation is of the form

$$\rho : F \to \operatorname{Aut}(N^{\mathbb{Q}}),$$

where $N^{\mathbb{Q}}$ is the radicable hull of N. The rational holonomy representation encodes some, but not all information of an almost-Bieberbach group as we discussed in Example 2.25. In this chapter we answer the question if the existence of an expanding map depends only on the rational holonomy representation.

The equivalent question for Anosov diffeomorphisms was already answered as described in Theorem 3.36. This chapter shows that a similar statement for the existence of an expanding map holds.

Theorem 6.15. *Let $\Gamma\backslash G$ be an infra-nilmanifold with associated rational holonomy representation $\rho : F \to \operatorname{Aut}(N^{\mathbb{Q}})$. The following statements are equivalent.*

(1) *The infra-nilmanifold $\Gamma\backslash G$ admits an expanding map.*

(2) *The Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ corresponding to $N^{\mathbb{Q}}$ has a positive grading, preserved by every automorphism in $\rho(F) \leq \operatorname{Aut}(N^{\mathbb{Q}}) \simeq \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$.*

(3) *There exists an expanding automorphism $\varphi \in \mathrm{Aut}(N^{\mathbb{Q}})$ which commutes with every element of $\rho(F)$.*

A similar criterion for the existence of a non-trivial self-cover on an infra-nilmanifold is given in Theorem 6.18. This answers a question of [7] where the author asks if the existence of a non-trivial self-cover on a nilmanifold depends only on the radicable hull.

Compared to the introduction of Part II, this chapter gives a solution to the gap between the injective group morphisms $\mathrm{Endo}(N)$ and the automorphisms of the radicable hull $\mathrm{Aut}(N^{\mathbb{Q}})$ in equation (5.1). The more general relation between $\mathrm{Endo}(N)$ and $\mathrm{Aut}(N^{\mathbb{Q}})$ is studied in the first two sections and these techniques are also fruitful for studying periodic points as we do in Chapter 7. These results are then applied to expanding maps and non-trivial self-covers in the following two sections. Finally we give many applications and examples in the last section.

6.1 Automorphisms of the radicable hull $N^{\mathbb{Q}}$

If $\varphi : N \to N$ is an injective group morphism, then Chapter 2 explained that φ has a unique extension $\varphi^{\mathbb{Q}} : N^{\mathbb{Q}} \to N^{\mathbb{Q}}$. The question we study in this section is which automorphisms $\varphi^{\mathbb{Q}} \in \mathrm{Aut}(N^{\mathbb{Q}})$ are the extension of a (necessarily injective) group morphism $\varphi : N \to N$ for some full subgroup N of $N^{\mathbb{Q}}$.

Question 6.1. Let $\varphi^{\mathbb{Q}} \in \mathrm{Aut}(N^{\mathbb{Q}})$ be an automorphism of a torsion-free nilpotent radicable group $N^{\mathbb{Q}}$. Under which conditions does there exist a full subgroup $N \leq N^{\mathbb{Q}}$ such that $\varphi^{\mathbb{Q}}$ induces an injective group morphism on N, i.e. such that $\varphi^{\mathbb{Q}}(N) \leq N$?

To answer this question, we first determine some properties of the extension $\varphi^{\mathbb{Q}} \in \mathrm{Aut}(N^{\mathbb{Q}})$. From Proposition 2.30 we know that φ induces an injective group morphism $\varphi^{\mathrm{lat}} : N^{\mathrm{lat}} \to N^{\mathrm{lat}}$ on the lattice hull N^{lat}. The group N is a finite index subgroup of its lattice hull N^{lat} and hence Proposition 2.23 shows that $N^{\mathbb{Q}}$ is the radicable hull of N^{lat} as well. The automorphism $\varphi^{\mathbb{Q}}$ is thus the unique extension of φ^{lat} to $N^{\mathbb{Q}}$.

The subset $\log(N^{\mathrm{lat}})$ is a finitely generated $\mathbb{Z}$-module in $\mathfrak{n}^{\mathbb{Q}}$ and thus there exists a basis for $\log(N^{\mathrm{lat}})$ as $\mathbb{Z}$-module. Every basis for $\log(N^{\mathrm{lat}})$ also forms a basis for the vector space $\mathfrak{n}^{\mathbb{Q}}$. By fixing a basis, we get an isomorphism from $\mathfrak{n}^{\mathbb{Q}}$ to $\mathbb{Q}^n$ as vector spaces such that this isomorphism maps $\log(N^{\mathrm{lat}})$ to $\mathbb{Z}^n$. The linear map $\varphi^{\mathbb{Q}}$ then maps $\mathbb{Z}^n$ into $\mathbb{Z}^n$ and this implies that the characteristic polynomial of $\varphi^{\mathbb{Q}}$ lies in $\mathbb{Z}[X]$.

The main conclusion of this section is that this last property exactly determines the automorphisms of $N^{\mathbb{Q}}$ which are the extension of some injective group morphism of a full subgroup of $N^{\mathbb{Q}}$.

Theorem 6.1. *Let $N^{\mathbb{Q}}$ be a radicable torsion-free nilpotent group and $\varphi^{\mathbb{Q}}$ an automorphism of $N^{\mathbb{Q}}$. There exists a full subgroup N of $N^{\mathbb{Q}}$ such that $\varphi^{\mathbb{Q}}(N) \leq N$ if and only if the characteristic polynomial of $\varphi^{\mathbb{Q}}$ lies in $\mathbb{Z}[X]$.*

This theorem is a consequence of the following more general statement which we need in the remaining part of this chapter.

Proposition 6.2. *Let $N^{\mathbb{Q}}$ be a radicable torsion-free nilpotent group with corresponding rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. Identify the group $\operatorname{Aut}(N^{\mathbb{Q}}) = \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ as a subgroup of $\operatorname{GL}(n, \mathbb{Q})$ by fixing a basis for $\mathfrak{n}^{\mathbb{Q}}$ as a vector space. There exists a full subgroup N of $N^{\mathbb{Q}}$ such that every automorphism of $N^{\mathbb{Q}}$ with integer entries induces an injective group morphism on N, i.e. if $\varphi \in \operatorname{Aut}(N^{\mathbb{Q}}) \leq \operatorname{GL}(n, \mathbb{Q})$ has integer entries, then $\varphi(N) \leq N$.*

Proof of Proposition 6.2. Denote the basis vectors for $\mathfrak{n}^{\mathbb{Q}}$ as $v_1, \ldots, v_n$ and take N_0 the full subgroup of $N^{\mathbb{Q}}$ generated by the elements $\exp(v_1), \ldots, \exp(v_n) \in N^{\mathbb{Q}}$. Since N_0 is an $\mathcal{F}$-group, its lattice hull $N = N_0^{\text{lat}}$ exists and we claim that N satisfies the properties of the proposition.

Let φ be an automorphism of $\operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ with integer entries in the basis $\{v_1, \ldots, v_n\}$, then we show that $\varphi(N) \leq N$. Since φ has integer entries, we have that $\varphi(N_0) \leq N = N_0^{\text{lat}}$, because $\varphi(v_i)$ is in the $\mathbb{Z}$-span of the basis $\{v_1, \ldots, v_n\}$ and N is a lattice group. From Proposition 2.30 it follows that

$$\varphi(N) = \varphi\left(N_0^{\text{lat}}\right) = \left(\varphi\left(N_0\right)\right)^{\text{lat}} \leq N^{\text{lat}}$$

and the latter is of course equal to N since N is a lattice group. We conclude that $\varphi(N) \leq N$. $\square$

From this proposition, it easily follows that every automorphism $\varphi^{\mathbb{Q}}$ with characteristic polynomial in $\mathbb{Z}[X]$ induces a group morphism on some full subgroup N of $N^{\mathbb{Q}}$.

Proof of Theorem 6.1. One implication was given in the discussion just before Theorem 6.1, namely if $\varphi^{\mathbb{Q}}$ is the extension of $\varphi : N \to N$, then its characteristic polynomial lies in $\mathbb{Z}[X]$.

For the other implication, Proposition 5.9 implies that there exists a basis for $\mathfrak{n}^{\mathbb{Q}}$ such that the matrix representation of $\varphi^{\mathbb{Q}}$ has integer entries. Fix such a basis for $\mathfrak{n}^{\mathbb{Q}}$ and apply Proposition 6.2 to this basis to find a full subgroup of $N^{\mathbb{Q}}$ such that $\varphi^{\mathbb{Q}}(N) \leq N$. This ends the proof. $\square$

For studying expanding maps, we are interested in automorphisms having only eigenvalues > 1 in absolute value. Starting from a positive grading, which is a decomposition

$$\mathfrak{n}^{\mathbb{Q}} = \bigoplus_{i>0} \mathfrak{n}_i^{\mathbb{Q}}$$

into subspaces $\mathfrak{n}_i^{\mathbb{Q}} \subseteq \mathfrak{n}^{\mathbb{Q}}$ such that $[\mathfrak{n}_i^{\mathbb{Q}}, \mathfrak{n}_j^{\mathbb{Q}}] \subseteq \mathfrak{n}_{i+j}^{\mathbb{Q}}$, Proposition 6.2 gives us a method for constructing many expanding automorphisms on at least one full subgroup of $N^{\mathbb{Q}}$.

Corollary 6.3. Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational nilpotent Lie algebra with a positive grading and $N^{\mathbb{Q}}$ the corresponding radicable nilpotent group. There exists a full subgroup N and some $k > 0$ such that for every prime p, there exists an injective group morphism $\varphi_p \in \mathrm{Endo}(N)$ with $\det(\varphi_p) = p^k$ which is also expanding. Moreover, these group morphisms φ_p commute with every automorphism that preserves the grading.

Proof. Fix a positive grading

$$\mathfrak{n}^{\mathbb{Q}} = \bigoplus_{i>0} \mathfrak{n}_i^{\mathbb{Q}}$$

for the rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ corresponding to $N^{\mathbb{Q}}$ and take a basis for $\mathfrak{n}^{\mathbb{Q}}$ such that every basis vector lies in a subspace $\mathfrak{n}_i^{\mathbb{Q}}$. For every prime p there exists an automorphism $\varphi_p : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ such that $\varphi_p(x) = p^i x$ for all $x \in \mathfrak{n}_i^{\mathbb{Q}}$.

From our choice of basis vectors, it follows that the matrix representation of φ_p in that basis has integer entries. Moreover φ_p commutes with every automorphism that preserves the grading. Every group morphism φ_p only has eigenvalues p^i for some $i > 0$ since the grading is positive and therefore φ_p is expanding. Finally, it holds that $\det(\varphi_p) = p^k$ for some fixed $k > 0$ depending only on the dimension of the subspaces $\mathfrak{n}_i^{\mathbb{Q}}$. The statement of the corollary now follows by taking an $\mathcal{F}$-group as in Proposition 6.2. □

In Corollary 6.3, the full subgroup N depends on the given positive grading of the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. The question that remains open is how we can construct expanding group morphisms on every full subgroup of $N^{\mathbb{Q}}$ and not just on this specific full subgroup N following from Proposition 6.2. We will answer this question in the following section, where it will be important to have these expanding group morphisms for every prime p.

6.2 Group morphisms of commensurable $\mathcal{F}$-groups

In this section, we study the relation between the sets $\mathrm{Endo}(N_1)$ and $\mathrm{Endo}(N_2)$ for commensurable $\mathcal{F}$-groups N_1 and N_2. The main result shows that most injective group morphisms $\varphi \in \mathrm{Endo}(N_1) \subseteq \mathrm{Aut}(N^{\mathbb{Q}})$ have some power φ^k such that $\varphi^k \in \mathrm{Endo}(N_2)$.

In the previous section we showed that if $\varphi^{\mathbb{Q}} \in \mathrm{Aut}(N^{\mathbb{Q}})$ is an automorphism with characteristic polynomial in $\mathbb{Z}[X]$, then there exists a full subgroup N of $N^{\mathbb{Q}}$ such that $\varphi^{\mathbb{Q}}(N) \leq N$. Of course, this does not mean that $\varphi^{\mathbb{Q}}$ will induce a group morphism on every full subgroup of $N^{\mathbb{Q}}$.

Example 6.4. The matrix

$$A = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

has characteristic polynomial $p(X) = X^2 - 3X + 1$ and thus there exists a full subgroup of $\mathbb{Q}^n$ such that the matrix A induces a morphism on this subgroup. An example of such a subgroup is the lattice spanned by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

But $\mathbb{Z}^2$ is also a full subgroup of $\mathbb{Q}^n$ and $A(\mathbb{Z}^2) \not\leq \mathbb{Z}^2$, so A doesn't induce a group morphism on $\mathbb{Z}^2$.

Let us assume that $\varphi : N \to N$ is an automorphism and consider the extension φ^{lat} of φ to the lattice hull N^{lat}. The morphism φ^{lat} is also an automorphism of N^{lat} by Proposition 2.30 and by fixing a basis for $\log(N^{\mathrm{lat}})$ as $\mathbb{Z}$-module, we get an isomorphism from $\mathfrak{n}^{\mathbb{Q}}$ to $\mathbb{Q}^n$ as vector spaces such that this isomorphism maps $\log(N^{\mathrm{lat}})$ to $\mathbb{Z}^n$. Since φ^{lat} is an automorphism, we get that under this isomorphism $\varphi^{\mathbb{Q}}(\mathbb{Z}^n) = \mathbb{Z}^n$. So every automorphism $\varphi \in \mathrm{Aut}(N)$ has $|\det(\varphi^{\mathbb{Q}})| = 1$. We conclude that for automorphisms $\varphi : N \to N$ the extension $\varphi^{\mathbb{Q}}$ is integer-like.

Vice versa, Theorem 6.1 and Proposition 2.30 imply that every integer-like automorphism of $\mathrm{Aut}(N^{\mathbb{Q}})$ induces an automorphism on some full subgroup of $N^{\mathbb{Q}}$. As Example 6.4 above shows, an integer-like automorphism of $N^{\mathbb{Q}}$ does not induce an automorphism on every full subgroup of $N^{\mathbb{Q}}$. The following result, see [28, Theorem 3.4.], shows that by taking some power of the automorphism, the statement does hold.

Theorem 6.5. *Let N be an $\mathcal{F}$-group with corresponding radicable hull $N^{\mathbb{Q}}$. If $\varphi \in \operatorname{Aut}(N^{\mathbb{Q}})$ is integer-like then there exists some $k > 0$ such that $\varphi^k(N) = N$.*

So if N_1 and N_2 are two full subgroups of $N^{\mathbb{Q}}$ and $\varphi : N_1 \to N_1$ an automorphism, then some power of φ induces an automorphism of N_2. For example, the third power of the matrix A above is equal to

$$A^3 = \begin{pmatrix} 17 & 4 \\ 4 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

This theorem is crucial for the study of Anosov diffeomorphisms on infra-nilmanifolds, as we will explain in Part III. It forms a crucial ingredient in the proof of Theorem 3.36.

Unfortunately, a similar theorem cannot be true in the more general case of automorphisms $\varphi \in \operatorname{Aut}(N^{\mathbb{Q}})$ with characteristic polynomial in $\mathbb{Z}[X]$, not even in the abelian case, as we can see from the following example.

Example 6.6. Consider the matrix

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$$

with characteristic polynomial $p(X) = X^2 - 3X + 2$ and take the vector $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. A computation shows that

$$B^k(v) = v + \frac{1}{2} \sum_{i=1}^{k} 2^{i-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and thus $B^k(v) \notin \mathbb{Z}^2$ for all $k > 0$. This shows us that the condition $|\det(\varphi)| = 1$ is necessary in Theorem 6.5.

To simplify notations, we will also denote by $\varphi \in \operatorname{Aut}(N^{\mathbb{Q}})$ the unique extension of an injective group morphism $\varphi : N \to N$. The main result of this section shows that we can partially restore Theorem 6.5 by putting constraints on the determinant of φ.

Theorem 6.7. *Let N_1 and N_2 be $\mathcal{F}$-groups with identical radicable hull $N^{\mathbb{Q}}$. Then there exists a finite number of primes $p_1, \ldots, p_l$ such that for every injective group morphism $\varphi : N_1 \to N_1$ with $p_j \nmid \det(\varphi)$ for all $j \in \{1, \ldots, l\}$, there exists some $k > 0$ such that $\varphi^k(N_2) \leq N_2$.*

Since the discussion above shows that every integer-like automorphism of $N^{\mathbb{Q}}$ induces an automorphism on some full subgroup of $N^{\mathbb{Q}}$, this theorem is really a generalization of Theorem 6.5.

The proof of this theorem uses a few technical lemmas. The first one gives us information about the index of the intersection of subgroups.

Lemma 6.8. *Let H be any group with finite index subgroups K_1 and K_2 of index respectively k_1 and k_2. If $\gcd(k_1, k_2) = 1$, then $K_1 \cap K_2$ is a subgroup of index k_1k_2 and if K_1' is a normal subgroup of index k_1 such that $K_1 \cap K_2 = K_1' \cap K_2$, then $K_1 = K_1'$.*

Proof. For the first statement, note that $[H : K_1 \cap K_2] = [H : K_1][K_1 : K_1 \cap K_2] = [H : K_2][K_2 : K_1 \cap K_2]$ and thus $k_j \mid [H : K_1 \cap K_2]$ for $j \in \{1, 2\}$. Since $\gcd(k_1, k_2) = 1$ and $[H : K_1 \cap K_2] \leq [H : K_1][H : K_2]$, we have that $[H : K_1 \cap K_2] = k_1k_2$. Note that this is equivalent to the fact that $[K_1 : K_1 \cap K_2] = k_2$.

For the second statement, we assume that $K_1 \cap K_2 = K_1' \cap K_2$ and we show that $K_1 \cap K_1'$ is a subgroup of index 1 in K_1. As K_1' is a normal subgroup, K_1K_1' is a subgroup of H. Since the index of $K_1 \cap K_1'$ in K_1 is equal to the index of K_1' in K_1K_1', it divides k_1. From the first statement we know that $K_1 \cap K_2 = K_1 \cap K_1' \cap K_2$ is a subgroup of index k_2 in K_1. Since $K_1 \cap K_1' \cap K_2 \leq K_1 \cap K_1' \leq K_1$, the index of $K_1 \cap K_1'$ in K_1 divides k_2. As the index of $K_1 \cap K_1'$ in K_1 divides both k_1 and k_2, we conclude from $gcd(k_1, k_2) = 1$ that it's equal to 1. □

Note that the first statement of Lemma 6.8 is equivalent to the fact that $K_1 \cap K_2$ has index k_1 in K_2 or $K_1 \cap K_2$ has index k_2 in K_1.

As a consequence of Lemma 6.8, we have the following result.

Lemma 6.9. *Let H be any group with a finite number of normal subgroups of index i and $\varphi : H \to H$ be an injective group morphism such that $\varphi(H)$ is a subgroup of finite index in H and $\gcd([H : \varphi(H)], i) = 1$. If we write the distinct normal subgroups of index i as $K_1, \ldots, K_m$, then there exists a permutation $\pi \in S_m$ such that $\varphi(K_j) = K_{\pi(j)} \cap \varphi(H)$ for all $j \in \{1, \ldots, m\}$.*

Proof. By the previous lemma, we know that $K_j \cap \varphi(H)$ is a normal subgroup of index i in $\varphi(H)$ and that if $K_j \neq K_{j'}$ also $K_j \cap \varphi(H) \neq K_{j'} \cap \varphi(H)$. Since $\varphi(H)$ is isomorphic to H, there are exactly m different normal subgroups of index i in $\varphi(H)$ and thus they are all of the form $K_j \cap \varphi(H)$. Since also $\varphi(K_1), \ldots, \varphi(K_m)$ are m distinct normal subgroup of index i in $\varphi(H)$, we find a permutation $\pi \in S_m$ such that $\varphi(K_j) = K_{\pi(j)} \cap \varphi(H)$. □

Note that the first condition of Lemma 6.9 is satisfied for every index i in a finitely generated group, see Theorem 11.1. Also, as we mentioned in Section 2.4, the image of an injective group morphism is always of finite index for these groups and the index is given by the absolute value of the determinant, see Proposition 2.30.

Take the notations of Lemma 6.9 and $\pi \in S_m$ the permutation corresponding to $\varphi : H \to H$ and some fixed index i. Then for any $k > 0$, we have that

$$\varphi^k(K_j) = \varphi^{k-1}\left(K_{\pi(j)} \cap \varphi(H)\right) = \varphi^{k-1}\left(K_{\pi(j)}\right) \cap \varphi^k(H)$$

since φ is injective and thus by induction we get that π^k is the permutation corresponding to φ^k

The last lemma we need is just the abelian version of Theorem 6.7. A uniform lattice of the vector space $\mathbb{Q}^n$ is a full subgroup of $\mathbb{Q}^n$ where we consider $\mathbb{Q}^n$ as the radicable hull of $\mathbb{Z}^n$.

Lemma 6.10. *Let L be a uniform lattice in $\mathbb{Q}^n$, then there exists primes $p_1, \ldots, p_l$ such that for every $A \in \mathrm{GL}(n, \mathbb{Q})$ with $A(\mathbb{Z}^n) \leq \mathbb{Z}^n$ and $p_i \nmid \det(A)$, there exists some $k > 0$ such that $A^k(L) \leq L$.*

Proof. Fix a basis for L as $\mathbb{Z}$-module (which also forms a basis for $\mathbb{Q}^n$ as vector space) and denote by P the matrix of change of basis from this basis to the standard basis. Take m the product of all the denominators of the entries of P and P^{-1} and take p_i all the primes dividing m. We claim that the primes p_i satisfy the conditions of the lemma.

So assume $A \in \mathrm{GL}(n, \mathbb{Q})$ with $A(\mathbb{Z}^n) \leq \mathbb{Z}^n$ and $p_i \nmid \det(A)$. The matrix representation of A^k for the chosen basis in L is given by $P^{-1}A^kP$ and thus we have to check that $P^{-1}A^kP$ has integer entries for some $k > 0$.

From the choice of the primes p_i, it follows that A projects to an element of $GL(n, \mathbb{Z}_m)$ and write this projection as $\pi(A)$. Since $\mathrm{GL}(n, \mathbb{Z}_m)$ is a finite group, there exists some $k > 0$ such that $\pi(A)^k = \pi(A^k) = \pi(I_n)$. This means that m divides every entry of $A^k - I_n$. So we have that

$$P^{-1}A^kP - I_n = P^{-1}(A^k - I_n)P$$

has integer entries because of our choice of m and thus also $P^{-1}A^kP$ has integer entries. $\square$

With the help of these lemmas, we are ready to give a proof of the general version of Theorem 6.7.

Proof of Theorem 6.7. Every injective group morphism $\varphi : N_1 \to N_1$ also induces an injective group morphism $\varphi^{\mathrm{lat}} : N_1^{\mathrm{lat}} \to N_1^{\mathrm{lat}}$ with the same determinant, so we can always assume that N_1 is a lattice group. By fixing a basis for $\log(N_1)$ as $\mathbb{Z}$-module (which is also a basis for the vector space $\mathfrak{n}^{\mathbb{Q}}$), the vector space $\mathfrak{n}^{\mathbb{Q}}$ is isomorphic to $\mathbb{Q}^n$ such that $\log(N_1)$ is equal to $\mathbb{Z}^n$ under this isomorphism.

Under this isomorphisms, the injective group morphisms of N_1 correspond to Lie algebra automorphisms which maps $\mathbb{Z}^n$ into $\mathbb{Z}^n$. If we assume that also N_2 is a lattice group, then $\log(N_2)$ is a uniform lattice in $\mathfrak{n}^{\mathbb{Q}} \simeq \mathbb{Q}^n$ and the theorem follows from Lemma 6.10.

Next we show that if the theorem is true for N_2, it is also true for every normal subgroup K of N_2. Let $p_1, \ldots, p_l$ be the finite number of primes corresponding to N_2 and add all the primes that divide the index of K in N_2. If $\varphi : N_1 \to N_1$ is a group morphism satisfying the conditions of the theorem, then we can assume $\varphi(N_2) \leq N_2$ by taking some power of φ. Since φ is an injective group morphism of N_2, we can apply Lemma 6.9. So take $K_1 = K, \ldots, K_m$ all normal subgroups of the same index as K in N_2 and by Lemma 6.9, we know that there exists $\pi \in S_m$ with $\varphi(K_j) = K_{\pi(j)} \cap \varphi(N_2)$. Now take $k > 0$ such that $\pi^k = 1$, then

$$\varphi^k(K) = \varphi^k(K_1) = K_{\pi^k(1)} \cap \varphi^k(N_2) = K_1 \cap \varphi^k(N_2) \leq K_1 = K$$

and therefore $\varphi^k(K) \leq K$ as we want.

For the proof in the general case, we know that $N_2 \leq N_2^{\mathrm{lat}}$ is a subgroup of finite index. Since N_2^{lat} is a nilpotent group, Theorem 2.15 implies that N_2 is a subnormal subgroup. This means we can find subgroups

$$H_0 = N_2 \leq H_1 \leq \ldots \leq H_m = N_2^{\mathrm{lat}}$$

with H_j normal in H_{j+1}. By applying the result of the previous paragraph on the subgroups H_j we conclude that the theorem also holds for N_2. $\square$

Remark 6.11. The proof of Theorem 6.7 also gives us a way of computing a set of primes as in the statement of the theorem. Let N_2 be a subgroup of index i in the lattice group N_2^{lat} and let P be the matrix of change of basis from N_1^{lat} to N_2^{lat} in the vector space $\mathfrak{n}^{\mathbb{Q}}$. Write $D = \frac{a}{b} \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$ for the determinant of P. Then the set of primes is given by all the prime divisors of the index i, the integer a and the denominators of the entries of P.

In Chapter 7 we start from an injective group morphism $\varphi : N_1 \to N_1$ and construct full subgroups N_2 such that the some power of φ induces a group morphism on N_2. Under some conditions, the points of N_2 will then project to periodic points.

For the particular case of integer-like automorphisms of $N^{\mathbb{Q}}$, the technicalities of Lemma 6.9 are avoided. So the proof of Theorem 6.5 is now immediate, avoiding the use of Jonquière groups as in the original paper [28].

Proof of Theorem 6.5. Let $\varphi : N_1 \to N_1$ be an automorphism. From Lemma 6.10 it follows that there exists some $k > 0$ such that $\varphi^k \in \operatorname{Aut}(N_2^{\mathrm{lat}})$. Since φ^k permutes the subgroups of index $[N_2^{\mathrm{lat}} : N_2]$ in N_2^{lat}, there is some power such that N_2 is mapped to itself. $\square$

From the simplified proof of Theorem 6.5, we can also compute the power k depending on the full subgroup N.

Example 6.12. The subgroup L generated by v_1, v_2 of Example 6.4 contains $\mathbb{Z}_2$ as a subgroup of index 2. Since L has three subgroups of index 2, the automorphism induced by A induces a permutation of S_3. Since the order of S_3 is 6, the matrix $A^6 \in \mathrm{GL}(2, \mathbb{Z})$. In fact, A induces a transitive permutation of S_3 and so already $A^3 \in \mathrm{GL}(2, \mathbb{Z})$.

6.3 Expanding maps on infra-nilmanifolds

In this section we combine the results of the previous sections to prove that the existence of an expanding map depends only on the rational holonomy representation. This describes an algebraic criterion for deciding whether an infra-nilmanifolds admits an expanding map.

Part of the proof is postponed to Chapter 8 where we study gradings on Lie algebras. We could already give the proof in this chapter as was done in the original paper [32], since it only uses basic techniques of algebraic number theory. But we prefer to bring together all techniques concerning gradings on Lie algebras in a separate chapter.

We start with the easier case of nilmanifolds, which follows almost directly now.

Theorem 6.13. *Let $N\backslash G$ be a nilmanifold with corresponding rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$, then the following statements are equivalent.*

(1) *The nilmanifold $N\backslash G$ admits an expanding map.*

(2) *The Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ has a positive grading.*

(3) *There exists an expanding automorphism $\varphi \in \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$.*

It was brought to our attention that part of this theorem was proved independently and by different techniques in [20, Theorem 1.13].

Proof. The fact that (1) implies (3) follows directly from Theorem 3.21. In Theorem 8.7 of Chapter 8 we will exactly show that (3) implies (2). So the only implication left for the theorem is (2) implies (1).

To show this implication, fix a positive grading of $\mathfrak{n}^{\mathbb{Q}}$ and a full subgroup N_0 of $N^{\mathbb{Q}}$ as in Corollary 6.3. Since N_0 and N have the same radicable hull, there exists primes $p_1, \dots, p_l$ as in Theorem 6.7. Now take p a prime such that $p \neq p_j$ for all $j \in \{1, \dots, l\}$ and the group morphism $\varphi_p : N_0 \to N_0$ as in Corollary 6.3. Because of our choice of p, there exists some power of φ_p such that $\varphi_p^k(N) \leq N$ and the induced map on $N\backslash G$ is an expanding map. $\square$

As a corollary, we see that the existence of an expanding map only depends on the commensurability class of the fundamental group.

Corollary 6.14. Let M_1 and M_2 be two nilmanifolds with commensurable fundamental groups. Then M_1 admits an expanding map if and only if M_2 admits an expanding map.

The proof is immediate since two $\mathcal{F}$-groups N_1 and N_2 are commensurable if and only if the radicable hulls $(N_1)^{\mathbb{Q}}$ and $(N_2)^{\mathbb{Q}}$ are isomorphic.

For infra-nilmanifolds, we have a similar theorem but with an extra condition originating from the rational holonomy group.

Theorem 6.15. *Let $\Gamma\backslash G$ be an infra-nilmanifold with associated rational holonomy representation $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$. Then the following statements are equivalent.*

(1) *The infra-nilmanifold $\Gamma\backslash G$ admits an expanding map.*

(2) *The Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ has a positive grading, preserved by every automorphism in $\rho(F) \leq \mathrm{Aut}(N^{\mathbb{Q}}) \simeq \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$.*

(3) *There exists an expanding automorphism $\varphi \in \mathrm{Aut}(N^{\mathbb{Q}})$ which commutes with every element of the holonomy group $\rho(F)$.*

Proof. The implication from (3) to (2) is again postponed to Chapter 8 and is exactly Theorem 8.8 from that chapter. We will first show that (1) implies (3) and conclude with the implication from (2) to (1).

(1) ⇒ (3) First assume that $\Gamma\backslash G$ admits an expanding map. Because of Theorem 3.21 and Theorem 2.26, we can assume that this expanding map is given by an affine infra-nilmanifold endomorphism $\bar{\alpha}$ with

$$\alpha = (g, \delta) \in N^{\mathbb{Q}} \rtimes_\rho \mathrm{Aut}(N^{\mathbb{Q}}).$$

It is easy to see that $\delta F \delta^{-1} = F$ and thus we can assume δ commutes with every element of F by replacing α by some power of α if necessary. The automorphism δ is then the expanding automorphism we need.

(2) ⇒ (1) Next assume that there exists a positive grading, preserved by every element of F. Let N be the Fitting subgroup of Γ and consider Γ as a subgroup of the semi-direct product $N^{\mathbb{Q}} \rtimes_\rho F$ where F is the rational holonomy group of Γ. This is possible because of the discussion on page 29. By Corollary 6.3, we know that there exists a full subgroup N_0 of $N^{\mathbb{Q}}$ with for every prime p an expanding group morphism $\varphi_p : N_0 \to N_0$ such that $\det(\varphi_p)$ is a power of p and φ_p commutes with every element of F.

Write $F = \{f_1, \ldots, f_l\}$ and fix elements $n_j \in N^{\mathbb{Q}}$ with $1 \leq j \leq l$ such that

$$(n_1, f_1), \ldots, (n_l, f_l) \in \Gamma,$$

where we consider $\Gamma \leq N^{\mathbb{Q}} \rtimes_\rho F$. So every element $\gamma \in \Gamma$ can be written as $\gamma = (n, 1)(n_j, f_j)$ for some $j \in \{1, \ldots, l\}$ and $n \in N$ and vice versa, if an element of $N^{\mathbb{Q}} \rtimes_\rho F$ can be written in this form, it is an element of Γ. Take N_1 the full subgroup of $N^{\mathbb{Q}}$ generated by N and the elements $n_1, \ldots, n_l$ and take N_2 a normal subgroup of finite index in N_1 such that $N_2 \leq N$. Now take $\varphi = \varphi_p : N_0 \to N_0$ with p different from all the primes we get by Theorem 6.7 for N_1, N_2 and N (where we take the first group equal to N_0 each time) and also $p \nmid [N_1 : N_2]$. By taking some power of φ we can thus assume that $\varphi(N_1) \leq N_1, \varphi(N_2) \leq N_2$ and $\varphi(N) \leq N$.

Consider now the group morphism that φ induces on ${}^{N_1}\!/\!_{N_2}$. We claim that this group morphism is injective (and thus an isomorphism since the group is finite). For this we have to show that $\varphi(N_1) \cap N_2 = \varphi(N_2)$. Since $N_2 \leq N_1$, we obviously have $\varphi(N_2) \leq \varphi(N_1) \cap N_2$, so it suffices to show that both subgroups have the same index in N_2. From Proposition 2.30 we know that $\varphi(N_2)$ is a subgroup of index $|\det(\varphi)|$ in N_2. Similarly, $\varphi(N_1)$ is a subgroup of index $|\det(\varphi)|$ in N_1 and by Lemma 6.8 and our choice of p, we get that $\varphi(N_1) \cap N_2$ is also a subgroup of index $|\det(\varphi)|$ in N_2. The claim thus follows.

By taking some power of φ, we can assume that φ induces the identity group morphism on ${}^{N_1}\!/\!_{N_2}$. Equivalently, we have for every n_j that $\varphi(n_j) = \tilde{n}_j n_j$ for some $\tilde{n}_j \in N_2 \leq N$. This implies that for every

$\gamma = (n,1)(n_j, f_j) \in \Gamma$, we have that

$$\begin{aligned}(1,\varphi)\gamma(1,\varphi^{-1}) &= (1,\varphi)(n,1)(n_j,f_j)(1,\varphi^{-1}) \\ &= (\varphi(n),1)(\varphi(n_j),\varphi f_j \varphi^{-1}) \\ &= (\varphi(n)\tilde{n}_j,1)(n_j,f_j) \in \Gamma\end{aligned}$$

since $\varphi(n)\tilde{n}_j \in N$. We conclude that $\varphi\Gamma\varphi^{-1} \leq \Gamma$ and thus φ induces an expanding map on the infra-nilmanifold $\Gamma\backslash G$.

□

6.4 Non-trivial self-covers on infra-nilmanifolds

An expanding map of an infra-nilmanifold $\Gamma\backslash G$ is an example of a non-trivial self-cover, i.e. a covering map $\Gamma\backslash G \to \Gamma\backslash G$ which is not a homeomorphism. So Theorem 6.13 and Theorem 6.15 give an algebraic way of constructing non-trivial self-covers on infra-nilmanifolds. The natural question we answer in this section is if there is an algebraic way of describing all infra-nilmanifolds which have a non-trivial self-cover. The following definition is then natural in this discussion.

Definition 6.16. We call a group H **cohopfian** if H contains no non-trivial subgroup isomorphic to itself. Equivalently, H is cohopfian if every injective group morphism $\varphi : H \to H$ is automatically surjective.

Every group morphism of an almost-Bieberbach group Γ induces a self-map on the infra-nilmanifold $\Gamma\backslash G$ by Theorem 2.26. Therefore infra-nilmanifolds with only trivial self-covers correspond to almost-Bieberbach groups which are cohopfian. Because of Proposition 2.30, an $\mathcal{F}$-group is cohopfian if and only if every injective group morphism has determinant ± 1.

Just as in Corollary 6.3, we have the following consequence of Proposition 6.2. This time the existence of a non-trivial and non-negative grading, so a decomposition

$$\mathfrak{n}^{\mathbb{Q}} = \bigoplus_{i \geq 0} \mathfrak{n}_i^{\mathbb{Q}}$$

with $[\mathfrak{n}_i^{\mathbb{Q}}, \mathfrak{n}_j^{\mathbb{Q}}] \subseteq \mathfrak{n}_{i+j}^{\mathbb{Q}}$ and $\mathfrak{n}_0^{\mathbb{Q}} \neq \mathfrak{n}^{\mathbb{Q}}$, implies the existence of many injective group morphisms which are not surjective on some full subgroup.

Corollary 6.17. Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational Lie algebra with non-negative and non-trivial grading. There exists a full subgroup N and a $k > 0$ such that for every prime p, there exists an injective group morphism $N \to N$ with determinant p^k. Moreover, this group morphism commutes with every automorphism that preserves the grading.

We deduce the following theorem, which is the analog of Theorem 6.15 for non-trivial self-covers. The proof is identical as before. Again part of the proof is postponed to Chapter 8 which groups all arguments about gradings on Lie algebras.

Theorem 6.18. *Let Γ be an almost-Bieberbach group with rational holonomy representation $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$. Then the following statements are equivalent.*

(1) *The infra-nilmanifold $\Gamma\backslash G$ admits a non-trivial self-cover;*

(2) *the group Γ is not cohopfian;*

(3) *the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ has a non-trivial and non-negative grading which is preserved by every element of $\rho(F) \leq \mathrm{Aut}(N^{\mathbb{Q}}) \simeq \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$;*

(4) *there exists an automorphism $\varphi \in \mathrm{Aut}(N^{\mathbb{Q}})$, commuting with every element of $\rho(F)$, with characteristic polynomial in $\mathbb{Z}[X]$ and $|\det(\varphi)| > 1$.*

We don't use the fact that the group Γ is torsion-free in the proof, so the same result is also true for the more general case of almost-crystallographic groups. Just as in case of Theorem 6.13, there is an independent proof by different techniques in [20, Theorem 1.12] for the case of nilmanifolds. The following is a consequence of Theorem 6.18.

Corollary 6.19. Let M_1 and M_2 be two nilmanifolds with commensurable fundamental groups. Then M_1 admits a non-trivial self-cover if and only if M_2 admits a non-trivial self-cover.

This answers a question of [7] about cohopfian $\mathcal{F}$-groups. In Chapter 8 we will combine Theorem 6.13 with some results about the existence of gradings on Lie algebras to answer even more questions from [7].

One of the consequences of this algebraic characterization is that an almost-Bieberbach group is cohopfian if its Fitting subgroup is cohopfian. We conclude this section by giving an elementary proof of this fact.

If H is any group, we call a subgroup $K \leq H$ injectively characteristic if $\varphi(K) \leq K$ for all injective group morphisms $\varphi : H \to H$. Every injectively characteristic subgroup is a normal subgroup, since every inner automorphism is injective. The notion of an injectively characteristic subgroup is weaker than a fully characteristic subgroup and stronger than a characteristic subgroup. It follows from Theorem 2.21 that the Fitting subgroup of an almost-Bieberbach group is an injectively characteristic subgroup which is in general not fully characteristic.

The following proposition shows why injectively characteristic subgroups are important, especially when studying cohopfian groups.

Proposition 6.20. *Let H be a group and $K \triangleleft H$ an injectively characteristic subgroup. If both K and ${}^{H}\!/_{K}$ are cohopfian then also H is cohopfian.*

Proof. Let $\varphi : H \to H$ be an injective group morphism of H. Since K is injectively characteristic, we know that $\varphi(K) \leq K$ and thus φ induces a injective morphism on K, which we call φ_K. Because K is cohopfian, we know that φ_K is an automorphism. Note that φ also induces a morphism on the group ${}^{H}\!/_{K}$ and call this induced map $\bar{\varphi}$. Thus we have the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "\varphi_K"] & H \arrow[r] \arrow[d, "\varphi"] & {}^{H}\!/_{K} \arrow[r] \arrow[d, "\bar{\varphi}"] & 1 \\ 1 \arrow[r] & K \arrow[r] & H \arrow[r] & {}^{H}\!/_{K} \arrow[r] & 1. \end{tikzcd}$$

As a consequence of the 5-lemma, it suffices to show that $\bar{\varphi}$ is an automorphism. Because ${}^{H}\!/_{K}$ is cohopfian, it is sufficient to show that $\bar{\varphi}$ is injective.

Assume that $\bar{\varphi}(hK) = K$ for some $h \in H$. This is equivalent to saying that $\varphi(h) \in K$. Since φ_K is surjective, there exists some $k \in K$ with $\varphi(h) = \varphi_K(k) = \varphi(k)$. From the injectivity of φ we have that $k = h$ and thus $hK = K$. This shows that $\bar{\varphi}$ is injective. We conclude that φ is an automorphism and thus H is cohopfian. □

By applying this proposition to the Fitting subgroup of an almost-Bieberbach group (and because finite groups are always cohopfian), we get the following:

Corollary 6.21. Let Γ be an almost-Bieberbach group with Fitting subgroup N. If N is cohopfian, then also Γ is cohopfian.

This elementary proof for Corollary 6.21 is also applicable in other situations. For example, the proof of Proposition 6.20 implies also the following proposition.

Proposition 6.22. *Let H be a group and $\varphi : H \to H$ an injective group morphism. If there exists a normal subgroup $N \triangleleft H$ such that $\varphi(N) = N$ and with ${}^{H}\!/_{N}$ cohopfian, then φ itself is an automorphism.*

Applied to the situation of affine infra-nilmanifold endomorphism, this leads to the following result.

Corollary 6.23. Let $\bar{\alpha}$ be an affine infra-nilmanifold endomorphism with determinant of α equal to ± 1, then $\bar{\alpha}$ is an affine infra-nilmanifold automorphism.

Proof. Let $\Gamma \backslash G$ be the infra-nilmanifold on which $\bar{\alpha}$ is defined. Since $\bar{\alpha}$ has determinant ± 1, $\bar{\alpha}$ induces an automorphism of the Fitting subgroup N of Γ. Applying Proposition 6.22 to N and using that finite groups are cohopfian then implies the statement. □

6.5 Examples and applications

In this section we highlight the most important applications of Theorem 6.15 and 6.18, which are the main results of this chapter. First we will give some explicit examples of Lie algebras corresponding to cohopfian $\mathcal{F}$-groups which are of minimal dimension or of minimal nilpotency class. Next we give a general way of constructing new examples starting from a linear algebraic $\mathbb{Q}$-group by using a result of [17].

Minimal dimension

In [7, Example 2.5.], the author gives an example of a 7-dimensional nilmanifold without self-covers and asks if this is an example of minimal dimension.

Question 6.2. Does every nilmanifold of dimension ≤ 6 admit a non-trivial self-cover?

The construction of [7, Example 2.5.] starts from a characteristically nilpotent Lie algebra. Recall that a nilpotent Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ is called characteristically nilpotent if every derivation of $\mathfrak{n}^{\mathbb{Q}}$ is a nilpotent endomorphism. Equivalently, the connected component of the identity in $\mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ consists only of unipotent elements. From Theorem 6.18 it follows that if the corresponding Lie algebra of an infra-nilmanifold is characteristically nilpotent, it can never admit a non-trivial self-cover. The following theorem gives a positive answer to Question 6.2.

Theorem 6.24. *All nilmanifolds of dimension ≤ 6 admit an expanding map.*

By Theorem 6.13 it suffices to show that all rational Lie algebras of dimension ≤ 6 have a positive grading and these Lie algebras have been classified, e.g. in [24]. In fact it suffices to check a small number of Lie algebras in this classification because we can already show that most Lie algebras of dimension ≤ 6 admit a positive grading. For example we have the following observation.

Proposition 6.25. *Every Lie algebra of nilpotency class* 1 *or* 2 *admits a positive grading*

Proof. For an abelian Lie algebra $\mathfrak{n}^{\mathbb{Q}}$, just take $\mathfrak{n}^{\mathbb{Q}} = \mathfrak{n}_1^{\mathbb{Q}}$ as positive grading. If $\mathfrak{n}^{\mathbb{Q}}$ is a nilpotent Lie algebra of nilpotency class 2, consider the subspace $\mathfrak{n}_2^{\mathbb{Q}} = \gamma_2(\mathfrak{n}^{\mathbb{Q}}) = [\mathfrak{n}^{\mathbb{Q}}, \mathfrak{n}^{\mathbb{Q}}]$. By taking any complementary subspace $\mathfrak{n}_1^{\mathbb{Q}}$ in $\mathfrak{n}^{\mathbb{Q}}$ for $\mathfrak{n}_2^{\mathbb{Q}}$, i.e. any subspace $\mathfrak{n}_1^{\mathbb{Q}}$ such that $\mathfrak{n}^{\mathbb{Q}} = \mathfrak{n}_1^{\mathbb{Q}} \oplus \mathfrak{n}_2^{\mathbb{Q}}$ as a vector space, we find a positive grading on $\mathfrak{n}^{\mathbb{Q}}$. $\square$

The results of [34] show that every 2-generated 4-step nilpotent Lie algebra and every 3-generated 3-step nilpotent Lie algebra has a positive grading as well. We conclude that only a few Lie algebras of the classification in [24] are left to check and we leave it as an exercise to construct positive gradings on these by hand.

Minimal nilpotency class

By Theorem 6.24 the minimal dimension of a nilmanifold admitting no non-trivial self-cover is 7. In [7, Example 2.5.], there is a 7-dimensional example of nilpotency class 6 and a natural question is for which nilpotency classes this minimal dimension can be obtained. We have the following example of nilpotency class 5.

Example 6.26. Let $\mathfrak{n}^{\mathbb{Q}}$ be the 5-step nilpotent rational Lie algebra with basis $X_1, X_2, \ldots, X_7$ and Lie bracket given by

$$\begin{array}{lll} [X_1, X_2] = X_3 & [X_1, X_5] = X_7 & [X_2, X_4] = X_7 \\ [X_1, X_3] = X_4 & [X_1, X_6] = X_7 & [X_2, X_5] = X_6 \\ [X_1, X_4] = X_7 & [X_2, X_3] = X_5 & [X_3, X_5] = X_7. \end{array}$$

A computation shows that $\mathfrak{n}^{\mathbb{Q}}$ is characteristically nilpotent, so every full subgroup of the corresponding radicable hull $N^{\mathbb{Q}}$ is cohopfian.

There is only a classification of complex Lie algebras of dimension 7 (see e.g. [79]), but no classification over the rational numbers $\mathbb{Q}$. In Chapter 8 we will show that a rational Lie algebra has a positive grading if and only if its complexification has a positive grading. Since all complex nilpotent Lie algebras of dimension 7 of nilpotency class 3 or 4 have a positive grading, this shows that Example 6.26 has minimal nilpotency class for dimension 7.

For nilpotency class 3 and 4, there are 8-dimensional examples of Lie algebras which are characteristically nilpotent, see [1]. The first example of a characteristically nilpotent Lie algebra in [45] was an example of nilpotency class 3 and dimension 8.

Non-trivial self-cover but no expanding map

Another interesting class of examples are Lie algebras without expanding automorphisms, but which do have non-negative and non-trivial grading. The minimal dimension of such an example is again 7, because of Theorem 6.24.

Example 6.27. Consider the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ with basis $X_1, \ldots, X_7$ and Lie bracket:

$$\begin{aligned} [X_1, X_2] &= X_3 & [X_2, X_3] &= X_5 & [X_2, X_6] &= X_7 \\ [X_1, X_3] &= X_4 & [X_2, X_4] &= X_6 & [X_3, X_5] &= -X_7 \\ [X_1, X_5] &= X_6 & [X_2, X_5] &= X_7. \end{aligned}$$

As an exercise, one can check that $\mathfrak{n}^{\mathbb{Q}}$ has no expanding automorphisms. But the map $\varphi : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ defined by

$$\begin{aligned} \varphi(X_1) &= X_1 & \varphi(X_5) &= 4X_5 \\ \varphi(X_2) &= 2X_2 & \varphi(X_6) &= 4X_6 \\ \varphi(X_3) &= 2X_3 & \varphi(X_7) &= 8X_7 \\ \varphi(X_4) &= 2X_4 \end{aligned}$$

is an automorphism of $\mathfrak{n}^{\mathbb{Q}}$ with characteristic polynomial in $\mathbb{Z}[X]$ and $\det(\varphi) = 2^{10} > 0$. The non-negative grading $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ corresponding to the automorphism φ is given by

$$\mathfrak{n}_0 = \langle X_1 \rangle$$

$$\mathfrak{n}_1 = \langle X_2, X_3, X_4 \rangle$$

$$\mathfrak{n}_2 = \langle X_5, X_6 \rangle$$

$$\mathfrak{n}_3 = \langle X_7 \rangle.$$

Again, since all 7-dimensional rational Lie algebras of nilpotency class 3 and 4 are graded, this is an example of minimal nilpotency class for dimension 7.

Remark 6.28. To check that the Lie algebras of Example 6.26 and Example 6.27 do not admit an expanding automorphism, one has to make some computations in the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. In Section 8.5 we show how the computations can be simplified by using the techniques of Chapter 8.

General construction

We present a general way of constructing nilmanifolds with various properties starting from an arbitrary $\mathbb{Q}$-group. Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational nilpotent Lie algebra and consider the projection map $\pi : \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}}) \to \operatorname{Aut}\left(\mathfrak{n}^{\mathbb{Q}} / [\mathfrak{n}^{\mathbb{Q}}, \mathfrak{n}^{\mathbb{Q}}]\right)$, with $\operatorname{Aut}\left(\mathfrak{n}^{\mathbb{Q}} / [\mathfrak{n}^{\mathbb{Q}}, \mathfrak{n}^{\mathbb{Q}}]\right) \approx \operatorname{GL}(n, \mathbb{Q})$ for some $n \in \mathbb{N}$. The kernel of π consists of unipotent automorphisms, see [55], and the image $\pi(\operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}}))$ forms the rational points $G(\mathbb{Q})$ of a linear algebraic $\mathbb{Q}$-group G of $\operatorname{GL}(n, \mathbb{C})$, as introduced in Section 5.3.

Let $\varphi \in \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ be an automorphism and denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $\pi(\varphi)$. Every eigenvalue μ of φ can then be written as an k-fold product $\lambda_{j_1} \ldots \lambda_{j_i}$ for some $k \in \{1, \ldots, c\}$, where c is the nilpotency class of $\mathfrak{n}^{\mathbb{Q}}$. Thus φ is an expanding automorphism if and only if $\pi(\varphi)$ is an expanding automorphism. So a nilmanifold admits an expanding map if and only if the group $G(\mathbb{Q}) = \pi(\operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}}))$ of the corresponding Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ has an expanding automorphism.

Similarly, the existence of a non-trivial self-cover is equivalent to the existence of an element in $G(\mathbb{Q})$ with characteristic polynomial in $\mathbb{Z}[X]$ and determinant > 1 in absolute value. Indeed, if $\pi(\varphi)$ has characteristic polynomial in $\mathbb{Z}[X]$, then all its eigenvalues are algebraic integers. Therefore also every eigenvalue of φ is an algebraic integer and thus the characteristic polynomial φ is an element of $\mathbb{Z}[X]$, see also Proposition 4.5.

Starting from a linear algebraic $\mathbb{Q}$-group G in $\mathrm{GL}(n, \mathbb{C})$ and $G(\mathbb{Q})$ its subgroup of rational points with $n \geq 2$, [17] shows that there always exists a rational nilpotent Lie algebra $\mathfrak{n}^{\mathbb{Q}}$, generated by n elements, such that $G(\mathbb{Q})$ is the image of the projection map π. So to construct nilmanifolds without non-trivial self-covers, it suffices to construct a linear algebraic $\mathbb{Q}$-group such that $G(\mathbb{Q})$ does not containing elements with characteristic polynomial in $\mathbb{Z}[X]$ and determinant > 1 in absolute value.

Example 6.29. Consider the linear algebraic $\mathbb{Q}$-group given by

$$G = \left\{ \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^{-1} \end{pmatrix} \;\middle|\; c \in \mathbb{C}^{\times} \right\}$$

and its subgroup of rational points

$$G(\mathbb{Q}) = \left\{ \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^{-1} \end{pmatrix} \;\middle|\; q \in \mathbb{Q}^{\times} \right\}.$$

Let $\mathfrak{n}^{\mathbb{Q}}$ a rational nilpotent Lie algebra with $G(\mathbb{Q})$ as image of the projection map π. The only elements in $G(\mathbb{Q})$ with characteristic polynomial in $\mathbb{Z}[X]$ are the ones corresponding to $q = \pm 1$, so every nilmanifold with Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ is cohopfian. Note that $\mathfrak{n}^{\mathbb{Q}}$ does have automorphisms with determinant $\neq 1$ in absolute value, so this is an example of a cohopfian $\mathcal{F}$-group which is different from all examples of [7].

Example 6.30. Consider the linear algebraic $\mathbb{Q}$-group

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \;\middle|\; c \in \mathbb{C}^{\times} \right\}.$$

with its subgroup $G(\mathbb{Q})$ of rational points given by

$$G(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \;\middle|\; q \in \mathbb{Q}^{\times} \right\}.$$

Let $\mathfrak{n}^{\mathbb{Q}}$ a rational nilpotent Lie algebra with H as image of the projection map π. Every element of H has eigenvalue 1, so $\mathfrak{n}^{\mathbb{Q}}$ doesn't admit an expanding automorphism. There does exists an automorphism with only eigenvalues in $\mathbb{Z}$ and determinant > 1 in absolute value, for example any automorphism φ with $\pi(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. So every nilmanifold with Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ has non-trivial self-covers but no expanding maps. In fact, the Lie algebra of Example 6.27 is an explicit example of a Lie algebra with image of π equal to $G(\mathbb{Q})$.

Remark 6.31. Every nilpotent Lie group $\mathfrak{n}^{\mathbb{Q}}$ comes with a linear algebraic $\mathbb{Q}$-group G as explained above. The group $G(\mathbb{Q})$ of rational points then contains all information about the existence of an expanding map or a partially expanding map on the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$.

To determine whether the nilmanifold corresponding to $\mathfrak{n}^{\mathbb{Q}}$ admits an Anosov diffeomorphism, it is not sufficient to know the group $G(\mathbb{Q})$ though. Of course, if $\mathfrak{n}^{\mathbb{Q}}$ has a hyperbolic integer-like automorphism then also $G(\mathbb{Q})$ has to contain such an automorphism, namely the image of this automorphism under the projection π. But it does not follow that every automorphism $\varphi \in \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ with $\pi(\varphi)$ hyperbolic and integer-like, is itself hyperbolic since the k-fold product of hyperbolic algebraic units can have absolute value 1.

Example 6.32. Let $\mathfrak{n}_1^{\mathbb{Q}} = \mathbb{Q}^2$ the abelian Lie algebra of dimension 2 and

$$\mathfrak{n}_2^{\mathbb{Q}} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \; x, y, z \in \mathbb{Q} \right\} \subseteq \mathbb{Q}^{3\times 3}$$

the Heisenberg Lie algebra as a subalgebra of the Lie algebra of matrices over $\mathbb{Q}$, see Example 2.2. The linear algebraic group G we get for both groups is $G = \mathrm{GL}(2, \mathbb{C})$ with $\mathrm{GL}(2, \mathbb{Q})$ as its subgroup of rational points. The Lie algebra $\mathfrak{n}_1^{\mathbb{Q}}$ corresponds to a nilmanifold admitting an Anosov diffeomorphism whereas the Lie algebra $\mathfrak{n}_2^{\mathbb{Q}}$ has no hyperbolic integer-like automorphisms, since $Z(\mathfrak{n}_2)^{\mathbb{Q}}$ is one-dimensional.

Example 6.32 shows the techniques of this section cannot be used to construct Anosov Lie algebras. In Chapter 9 we will construct a nilmanifold admitting an Anosov diffeomorphism but no expanding map. The Lie algebra corresponding to this nilmanifold has to be constructed explicitly to guarantee the existence of an Anosov.

Chapter 7

Periodic points of affine infra-nilmanifold endomorphisms

In this chapter we study the periodic and eventually periodic points of affine infra-nilmanifold endomorphisms. This chapter is a bit different from the other chapters in Part II, since it studies general affine infra-nilmanifold endomorphisms and not only the expanding ones. The reason is that the methods used for studying periodic points of these self-maps are very closely related to the methods of the previous chapter for studying expanding maps.

In the previous chapter we started from two commensurable $\mathcal{F}$-groups N_1 and N_2 and studied the relation between their injective group morphisms. One of the main results, namely Theorem 6.7, gave a condition under which an injective group morphism $\varphi \in \mathrm{Endo}(N_1)$ has some power φ^k such that $\varphi^k \in \mathrm{Endo}(N_2)$. For studying periodic points, we start from an $\mathcal{F}$-group N and a fixed group morphism $\varphi : N \to N$ which has a unique extension $N^{\mathbb{Q}} \to N^{\mathbb{Q}}$ which we also denote by φ for simplicity. A point $n \in N^{\mathbb{Q}}$ will be periodic if and only if $\varphi^k(n)n^{-1} \in N$ for some $k > 0$. This corresponds more or less to finding a $k > 0$ such that φ^k induces a group morphism on the full subgroup of $N^{\mathbb{Q}}$ generated by n and N. In this setting, the results of the previous chapter are important for finding full subgroups on which some power of φ induces a group morphism.

This chapter continues the work that was started in [57]. In that paper, the authors show that the set of eventually periodic points of an infra-nilmanifold endomorphism forms a dense subset of the infra-nilmanifold. We extend their results in several different directions.

First of all, we study the more general class of affine infra-nilmanifold endomorphisms in this chapter. This means that our maps are not necessarily induced by an automorphism of the covering Lie group, but by an affine transformation. In Section 7.1, we show how the study of this more general class can be reduced to the situation of nilmanifold endomorphisms.

Next, we prove that not only the set of eventually periodic points, but also the smaller set of periodic points is a dense subset of the infra-nilmanifold. This result is stated in Theorem 7.14. The main idea is to avoid using the Baker-Campbell-Hausdorff formula, since this technique from [57] is not applicable for all nilmanifold endomorphisms.

Finally we give a full description of the set of eventually periodic points for such a self-map in Theorem 7.18. This improves the results of [57] in the sense that this paper only constructs a subset of eventually periodic points which is dense in the manifold. The main tool for this theorem is to look for a necessary condition for a point of the infra-nilmanifold to be (eventually) periodic.

7.1 Reduction to nilmanifold endomorphisms

In this section, we show how the study of (eventually) periodic points of affine infra-nilmanifold endomorphisms can be reduced to the more restrictive case of nilmanifold endomorphisms. These type of maps are induced by automorphisms on nilmanifolds and are therefore easier to handle. Their (eventually) periodic points are then studied in more detail in the following sections.

We start with the following lemma which, although it is very elementary, forms an important ingredient for the main results of this chapter.

Lemma 7.1. *Let $f : S \to S$ be a map on a finite set S. Then we have* $\mathrm{Per}(f) \neq \varnothing$ *and* $\mathrm{ePer}(f) = S$. *If f is moreover injective then* $\mathrm{Per}(f) = S$.

Proof. Take any $s \in S$ and consider its orbit

$$\left\{ f^k(s) \mid k > 0 \right\} \subseteq S.$$

Since S is finite, there exists $k_1 < k_2$ such that $f^{k_1}(s) = f^{k_2}(s)$. This implies that $f^{k_1}(s)$ is a periodic point of f and thus $\mathrm{Per}(f) \neq \varnothing$. The element $s \in \mathrm{ePer}(f)$ as well and since s was chosen arbitrary, this implies $\mathrm{ePer}(f) = S$. If f is injective, then f forms a permutation of the finite set S, hence f has finite order and $\mathrm{Per}(f) = S$. $\square$

To main idea of the chapter is to construct finite subsets of the infra-nilmanifold which are preserved by the affine infra-nilmanifold endomorphism. In this way, Lemma 7.1 helps us constructing (eventually) periodic points. The following proposition shows that Lemma 7.1 allows us to study the behavior of (eventually) periodic points under finite covering maps as well.

Proposition 7.2. *Let $p : E \to B$ a finite covering map and consider maps $f : E \to E$ and $g : B \to B$ such that f is a lift of g. Then the following statements hold.*

(1) $p^{-1}(\mathrm{ePer}(g)) = \mathrm{ePer}(f)$.

(2) $p(\mathrm{Per}(f)) = \mathrm{Per}(g)$.

(3) *If E is a regular covering map and*

$$f_* : {}^{\pi_1(B)}\!\big/_{\pi_1(E)} \to {}^{\pi_1(B)}\!\big/_{\pi_1(E)}$$

is injective, then $p^{-1}(\mathrm{Per}(g)) = \mathrm{Per}(f)$.

Proof. For the first statement, the inclusion $\mathrm{ePer}(f) \subseteq p^{-1}(\mathrm{ePer}(g))$ follows immediately from the fact that f is a lift of the map g. For the other inclusion, take any $b \in \mathrm{ePer}(g)$, then there exists $n, k > 0$ such that $g^{n+k}(b) = g^n(b)$. For every $e \in p^{-1}(g^n(b))$, we have that

$$p(f^k(e)) = g^k(p(e)) = g^k(g^n(b)) = g^n(b) = p(e),$$

and thus f^k preserves the finite set $p^{-1}(g^n(b))$. By Lemma 7.1 we conclude that every point of $p^{-1}(g^n(b))$ is eventually periodic for the map f^k and thus also for the map f. Since every $e \in p^{-1}(b)$ satisfies $f^n(e) \in p^{-1}(g^n(b))$, this implies that $p^{-1}(b) \subseteq \mathrm{ePer}(f)$.

The inclusion $p(\mathrm{Per}(f)) \subseteq \mathrm{Per}(g)$ of the second statement follows again from the definition of lift of a map. For the other inclusion, take $b \in \mathrm{Per}(g)$ and $k > 0$ such that $g^k(b) = b$. Similarly as above, f^k preserves the finite set $p^{-1}(b)$ and by Lemma 7.1, we conclude that there exists $e \in p^{-1}(b)$ such that $e \in \mathrm{Per}(f^k) = \mathrm{Per}(f)$ or thus that $\mathrm{Per}(g) \subseteq p(\mathrm{Per}(f))$.

For the last statement, we only have to show that f^k is injective on $p^{-1}(b)$ in the previous argument. Denote by H the finite group of deck transformations of p which is isomorphic to the group ${}^{\pi_1(B)}\!\big/_{\pi_1(E)}$. So the map f and therefore also f^k induces an injective group morphism $f_* : H \to H$. Fix an element $e_0 \in p^{-1}(b)$, then every other element $e \in p^{-1}(b)$ is uniquely represented as

$e = {}^{h}e_0$ for some $h \in H$. Under the identifications explained in Section 2.2 we have that

$$f^k(e) = {}^{f_*^k(h)}f^k(e_0).$$

This implies that f^k is injective on the set $p^{-1}(b)$ and thus every $e \in p^{-1}(b)$ is periodic. $\square$

There is an important difference between statements (1) and (2) of Proposition 7.2, which we illustrate with the following example.

Example 7.3. Consider the torus $B = \mathbb{Z}^n \backslash \mathbb{R}^n$ and take integers $k_1, \dots, k_n$ with $|\prod_{i=1}^n k_i| > 1$. Let δ be the linear map $\mathbb{R}^n \to \mathbb{R}^n$ induced by the matrix

$$\begin{pmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_n \end{pmatrix} \in \mathrm{GL}(n, \mathbb{Q}).$$

Consider the lattice $L = \delta(\mathbb{Z}^n) \leq \mathbb{Z}^n$ of $\mathbb{R}^n$ and denote the torus $L \backslash \mathbb{R}^n$ as E. Since L is a finite index normal subgroup of $\mathbb{Z}^n$, E is a regular finite cover of B and take $p : E \to B$ the covering map. It holds that $\delta(\mathbb{Z}^n) \leq \mathbb{Z}^n$ and $\delta(L) \leq \delta(\mathbb{Z}^n) = L$ and thus δ induces a nilmanifold endomorphism on both nilmanifolds B and E. We denote these induced nilmanifold endomorphisms as $f : E \to E$ and $g : B \to B$. Note that f is indeed a lift of the map g.

Denote by q the universal covering map $\mathbb{R}^n \to E$, then $p \circ q$ is the universal covering map for B. Every point of $X = q(\mathbb{Z}^n)$ is an eventually periodic point of f, since

$$f^m(q(z)) = q(\delta^m(z)) \in q(L) = \{eL\}$$

for every $m > 0$ and $z \in \mathbb{Z}^n$. But only the point $q(0) \in X$ is a periodic point of the map f. The set X has $\det(\delta) = |\prod_{i=1}^n k_i| > 1$ elements and so not every point in $p^{-1}(\mathrm{Per}(g)) \supseteq X$ is periodic, although $p(\mathrm{Per}(f)) = \mathrm{Per}(g)$.

By applying this result to affine infra-nilmanifold endomorphisms we get the following consequence of Proposition 7.2.

Theorem 7.4. *Let $\bar{\alpha}$ be an affine infra-nilmanifold endomorphism on the infra-nilmanifold $\Gamma \backslash G$ induced by $\alpha \in \mathrm{Aff}(G)$. Let $N \lhd \Gamma$ be the Fitting subgroup of Γ and consider the natural covering map $p : N \backslash G \to \Gamma \backslash G$. Then α induces an affine nilmanifold endomorphism $\tilde{\alpha}$ on $N \backslash G$ which is the lift of $\bar{\alpha}$ and the (eventually) periodic points satisfy*

$$p^{-1}\left(\mathrm{ePer}(\bar{\alpha})\right) = \mathrm{ePer}(\tilde{\alpha})$$

$$p^{-1}\left(\mathrm{Per}(\bar{\alpha})\right) = \mathrm{Per}(\tilde{\alpha}).$$

Proof. We identify the fundamental group in a point Γg_0 with the group Γ of deck transformations as in Section 2.2. Since $\alpha : G \to G$ is a diffeomorphism, we know that $\alpha_*(\gamma) = \alpha\gamma\alpha^{-1}$ for all $\gamma \in \Gamma$. Write $\alpha = (g, \delta)$, then

$$\alpha N \alpha^{-1} = g\delta(N)g^{-1} \subseteq \Gamma \cap G = N$$

and thus α induces an affine nilmanifold endomorphism on the finite covering space.

The holonomy group $\Gamma\diagup N = F$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$ in a natural way. The group morphism induces by α_* on F is given by $\delta f \delta^{-1}$ for all $f \in F$ under this isomorphism. Therefore this map is injective and thus the statement of the theorem follows from Proposition 7.2. □

This theorem reduces the study of periodic points of affine infra-nilmanifold endomorphisms to the study of affine nilmanifold endomorphisms. Since we are interested in the density of (almost) periodic points, the following lemma is important.

Lemma 7.5. *Let $p : E \to B$ a covering map and consider subsets $X \subseteq E$ and $Y \subseteq B$. Then the following statements are true.*

(1) *If X is dense in E, then $p(X)$ is dense in B.*

(2) *Y is dense in B if and only if $p^{-1}(Y)$ is dense in E.*

Note that it is not true that X is dense in E if $p(X)$ is dense in B. For example, if $p : \mathbb{R} \to \mathbb{Z}\backslash\mathbb{R}$ is the natural covering map, then $p([0, 1])$ is dense in $\mathbb{Z}\backslash\mathbb{R}$ but $[0, 1]$ is not dense in $\mathbb{R}$.

Proof. The first statement is true since p is a continuous surjective map. For the second statement, take an open set $U \subseteq E$ and consider the subset $p(U) \subseteq B$ which is also open since p is a local homeomorphism. Since Y is dense, there exists $y \in Y \cap p(U)$ and take $u \in U$ such that $p(u) = y$. Because $u \in p^{-1}(Y) \cap U$, this finishes the proof. □

Not every affine infra-nilmanifold endomorphism has periodic points, as we can see from the following example.

Example 7.6. Consider the torus $\mathbb{Z}^n\backslash\mathbb{R}^n$ and the affine infra-nilmanifold endomorphism induced by the affine map $\alpha = (b, I_n) \in \mathrm{Aff}(\mathbb{R}^n)$. Note that

$$\bar{\alpha}^k(\mathbb{Z}^n + x) = \mathbb{Z}^n + x + kb$$

and thus $\mathbb{Z}^n + x$ is a periodic point if and only if $kb \in \mathbb{Z}^n$ for some $k > 0$. This implies that the map $\bar{\alpha}$ has periodic points if and only if $b \in \mathbb{Q}^n$. By taking $b \in \mathbb{R}^n \setminus \mathbb{Q}^n$, this gives us a class of affine infra-nilmanifold endomorphisms without periodic points.

Example 7.6 shows that there are affine infra-nilmanifold endomorphisms which have no periodic points at all. If an affine nilmanifold endomorphism has a periodic point, we can further reduce the problem to nilmanifold endomorphisms by the following result.

Theorem 7.7. *Let $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ be an affine infra-nilmanifold endomorphism induced by the affine transformation $\alpha \in \mathrm{Aff}(G)$. Then $\bar{\alpha}$ is topologically conjugate to an infra-nilmanifold endomorphism if and only if $\bar{\alpha}$ has a fixed point.*

The proof of Theorem 7.7 is identical to the proof of [30, Theorem 4.5.] which states that every expanding affine infra-nilmanifold endomorphisms is topologically conjugate to an expanding infra-nilmanifold endomorphism. We do give the proof here, since it also gives an exact form of the topologically conjugacy between these maps.

Proof. One direction of Theorem 7.7 is clear, since every nilmanifold endomorphism has a fixed point and a topological conjugation maps fixed points to fixed points by Proposition 3.2.

For the other direction, write $\alpha = (g, \delta)$ and let Γg_0 be a fixed point of $\bar{\alpha}$. Identify the Lie group G with the subgroup of pure translations in $\mathrm{Aff}(G)$. Consider the subgroup $\Gamma' = g_0^{-1}\Gamma g_0 \leq \mathrm{Aff}(G)$ which is also an almost-Bieberbach group modeled on the Lie group G. The point Γg_0 is a fixed point, so $\delta(g_0) \in g^{-1}\Gamma g_0$ or equivalently $g_0^{-1} g\delta(g_0) \in \Gamma'$. Since $\alpha\Gamma\alpha^{-1} = g\delta\Gamma\delta^{-1}g^{-1} \leq \Gamma$, we have that $\delta\Gamma\delta^{-1} \leq g^{-1}\Gamma g$.

Consider the homeomorphism $h : \Gamma\backslash G \to \Gamma'\backslash G$ given by

$$h(\Gamma g) = \Gamma' g_0^{-1} g.$$

This map is indeed well-defined, since if $\gamma \in \Gamma$, then $\gamma' = g_0^{-1}\gamma g_0 \in \Gamma'$ and we have that

$$g_0^{-1}\gamma g = \gamma' g_0^{-1} g \in \Gamma' g_0^{-1} g.$$

The automorphism δ induces an infra-nilmanifold endomorphism on $\Gamma'\backslash G$ since

$$\delta\Gamma'\delta^{-1} = \delta(g_0^{-1})\delta\Gamma\delta^{-1}\delta(g_0) = g_0^{-1}\Gamma g g^{-1}\Gamma g g^{-1}\Gamma g_0 \leq g_0^{-1}\Gamma g_0.$$

Moreover, for every $\Gamma' x \in \Gamma' \backslash G$, we have that

$$h\left(\bar{\alpha}\left(h^{-1}(\Gamma' x)\right)\right) = h(\bar{\alpha}(\Gamma g_0 x)) = h(\Gamma g \delta(g_0 x)) = \Gamma' g_0^{-1} g \delta(g_0)\delta(x) = \Gamma' \delta(x)$$

and so $h\bar{\alpha}h^{-1} = \bar{\delta}$. Thus h is a topological conjugacy between $\bar{\alpha}$ and $\bar{\delta}$. □

Remark 7.8. The proof of Theorem 7.7 also gives us the exact form of a possible topological conjugation between $\bar{\alpha}$ and $\bar{\delta}$. For the remaining part of this chapter, we are mainly interested in the lift of this homeomorphism to the universal cover G and the subgroup N of pure translations.

If Γg_0 is the fixed point of $\bar{\alpha}$, then there is a topological conjugacy which maps Γg_0 to $\Gamma' e$ with lift to the universal cover G given by $L_{g_0^{-1}}$. If N and N' are the subgroups of pure translations of Γ and Γ' respectively, then $N' = g_0^{-1} N g_0$ and thus

$$(N')^{\mathbb{Q}} = g_0^{-1} N^{\mathbb{Q}} g_0.$$

In the next sections we use this fact to describe the (eventually) periodic points of an affine infra-nilmanifold endomorphism starting from the description for nilmanifold endomorphisms.

In the following section, we compute $\mathrm{Per}(\bar{\delta})$ and $\mathrm{ePer}(\bar{\delta})$ for nilmanifold endomorphisms $\bar{\delta}$. The results of this section thus allow us to interpret these results in terms of $\mathrm{Per}(\bar{\alpha})$ and $\mathrm{ePer}(\bar{\alpha})$ from general affine infra-nilmanifold endomorphisms.

7.2 Sufficient condition for (eventually) periodic points

In this section, we give a sufficient condition for points of a nilmanifold to be periodic, based on the relative order of an element in the radicable hull. We first define this relative order and show the relation with the index of a subgroup for nilpotent groups.

Let G be a group and take $H \leq G$ any subgroup of finite index. We define the relative order of an element $g \in G$ with respect to the subgroup H as

$$\mathrm{ord}_H(g) = \min\{n > 0 \mid g^n \in H\}.$$

Note that $\mathrm{ord}_H(g)$ does not depend on the overlying group G, therefore we avoid using the group G in the notation. If N is a normal subgroup with natural projection map $p : G \to {}^{G}\!/_{N}$, then $\mathrm{ord}_N(g)$ is the order of the element $p(g)$

in ${}^{G}\!/_{N}$. From the definition, it follows that $g^n \in H$ if and only if $\operatorname{ord}_H(g) \mid n$. Let $H_1 \leq H_2 \leq G$ be subgroups, then by definition $g^{\operatorname{ord}_{H_1}(g)} \in H_1 \leq H_2$ and thus $\operatorname{ord}_{H_2}(g) \mid \operatorname{ord}_{H_1}(g)$.

Let $s > 0$ be any integer and consider the set

$$X_s = \{g \in G \mid \operatorname{ord}_H(g) \mid s\}.$$

The set X_s does not form a subgroup of G in general and denote by $H^{\frac{1}{s}}$ the subgroup generated by X_s. If $\varphi : G \to G$ is a group morphism such that $\varphi(H) \leq H$, then

$$\varphi(g)^{\operatorname{ord}_H(g)} = \varphi(g^{\operatorname{ord}_H(g)}) \in \varphi(H) \leq H$$

and thus $\operatorname{ord}_H(\varphi(g)) \mid \operatorname{ord}_H(g)$. This implies that $\varphi(X_s) \subseteq X_s$ and thus φ induces a group morphism on every subgroup $H^{\frac{1}{s}}$ for $s > 0$.

If N is a $\mathcal{F}$-group with radicable hull $N^{\mathbb{Q}}$, then every element $n \in N^{\mathbb{Q}}$ lies in a subgroup which has N as a finite index subgroup. Therefore, we can define $\operatorname{ord}_N(g)$ for every element $g \in N^{\mathbb{Q}}$. The group N is always a subgroup of finite index of the group $N^{\frac{1}{s}}$ in this case. The advantage of working with these groups $N^{\frac{1}{s}}$ is that φ always induces a group morphism on them. This avoids using Theorem 6.7 for these full subgroups.

By using this relative order, we can construct eventually periodic points for every nilmanifold endomorphism.

Theorem 7.9. *Let N be a lattice of the nilpotent Lie group G with radicable hull $N^{\mathbb{Q}}$. If $\bar{\delta} : N\backslash G \to N\backslash G$ is a nilmanifold endomorphism then*

$$p(N^{\mathbb{Q}}) \subseteq \operatorname{ePer}(\bar{\delta}).$$

Proof. Take $n \in N^{\mathbb{Q}}$ arbitrary and let $s = \operatorname{ord}_N(n)$, which implies that $n \in N^{\frac{1}{s}}$. Since N is a finite index subgroup of $N^{\frac{1}{s}}$, the set $p(N^{\frac{1}{s}})$ is finite. From the previous discussion we know that $\delta(N^{\frac{1}{s}}) \leq N^{\frac{1}{s}}$ because $\delta(N) \leq N$. This implies that $\bar{\delta}$ induces a map on the finite set $p(N^{\frac{1}{s}})$ and thus every point of $p(N^{\frac{1}{s}})$ is eventually periodic. In particular, $p(n) \in p(N^{\frac{1}{s}})$ is eventually periodic. □

To construct periodic points, we need to know that the map $\bar{\delta}$ is injective on the set $p\left(N^{\frac{1}{s}}\right)$. Therefore we need a more careful study of the groups $N^{\frac{1}{s}}$. So although we know that φ always induces a group morphism on $N^{\frac{1}{s}}$, we still need the techniques of Chapter 6 to show that it induces an injective map on $p\left(N^{\frac{1}{s}}\right)$.

First we determine the relation between the index of a subgroup and the relative order. For normal subgroups N of G, it follows from Cauchy's theorem that if $p \mid [G : N]$, then there always exists an element $g \in G$ such that $\mathrm{ord}_N(G) = p$. We show that this is also true for subnormal subgroups H of finite index.

Lemma 7.10. *Let G be a group and $H \leq G$ a subnormal subgroup of finite index. If $p \mid [G : H]$, then there exists $g \in G$ such that*

$$\mathrm{ord}_H(g) = p.$$

In particular, we can apply this lemma for every nilpotent group G, since all subgroups of a nilpotent group are subnormal, see Theorem 2.15.

Proof. The subgroup H is subnormal in G, meaning that there exists a sequence of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_m = G$$

such that H_i is normal in H_{i+1}. Since $p \mid [G : H]$, there exists an i such that $p \mid [H_{i+1} : H_i]$ and by Cauchy's theorem we know that there exists $g \in H_{i+1}$ such that $p = \mathrm{ord}_{H_i}(g)$. Because $H \leq H_i$, we have that $p = \mathrm{ord}_{H_i}(g) \mid \mathrm{ord}_H(g)$ and thus a power of the element g satisfies the condition of the lemma. □

So information about the relative order gives us information about the index of the subgroup. The following result of [93, Proposition 6.3] gives us information about $\mathrm{ord}_N(n)$ for elements $n \in N^{\frac{1}{s}}$.

Proposition 7.11. *Let G be a nilpotent group of nilpotency class at most c and take any $s \in \mathbb{N}$. Let X be a set of generators of G and put $H = \langle x^s \mid x \in X \rangle$. Then*

$$G^{s^m} \leq H$$

for $m = \frac{1}{2}c(c+1)$.

Note that the result can be translated to relative orders of subgroups, since

$$G^{s^m} \leq H$$

is equivalent to

$$\mathrm{ord}_H(g) \mid s^m$$

for all $g \in G$. If we apply this result to the groups $N^{\frac{1}{s}}$, we get the following translation.

Proposition 7.12. *Let G be a finitely generated nilpotent group and $H \leq G$ a subgroup. Then $[H^{\frac{1}{s}} : H] \mid s^k$ for some $k \in \mathbb{N}$.*

Proof. Apply Proposition 7.11 to the set of generators

$$X_s = \{g \in G \mid \mathrm{ord}_H(g) \mid s\}$$

of $H^{\frac{1}{s}}$. The group generated by the elements x^s for $x \in X$ is a subgroup of H, which we denote by H'.

Suppose that $[H^{\frac{1}{s}} : H] \nmid s^k$ for every $k > 0$ or equivalently that there exists a prime p with $\gcd(p, s) = 1$ and $p \mid [H^{\frac{1}{s}} : H]$. Then also $p \mid [H^{\frac{1}{s}} : H']$ and because of Lemma 7.10 we get that there exists a $g \in H^{\frac{1}{s}}$ such that $\mathrm{ord}_{H'}(g) = p$. This is a contradiction to Proposition 7.11. □

To construct periodic points for a nilmanifold endomorphism, we start from the idea of Theorem 7.9 in which we constructed eventually periodic points. Following Lemma 7.1, the only thing we still need to find periodic points is injectivity of the induced map. This idea is exploited in the following theorem by restricting to certain groups $N^{\frac{1}{s}}$ depending on the determinant of the endomorphism.

Theorem 7.13. *Let $N\backslash G$ be a nilmanifold and take the radicable hull $N^{\mathbb{Q}}$ of N. Consider the subset*

$$N_D^{\mathbb{Q}} = \left\{n \in N^{\mathbb{Q}} \mid \gcd(D, \mathrm{ord}_N(g)) = 1\right\}.$$

If $\bar{\delta}$ is a nilmanifold endomorphism on $N\backslash G$ with determinant D, then

$$p(N_D^{\mathbb{Q}}) \subseteq \mathrm{Per}(\bar{\delta}).$$

Proof. Take $n \in N_D^{\mathbb{Q}}$, so $\mathrm{ord}_N(n) = s$ with $\gcd(D, s) = 1$. Consider the group $N^{\frac{1}{s}}$ which contains N as a subgroup of finite index and note that $n \in N^{\frac{1}{s}}$. We claim that $\bar{\delta}$ is injective on the finite set $p(N^{\frac{1}{s}})$. Because of Lemma 7.1, this implies that the point $p(n)$ is periodic.

Injectivity of $\bar{\delta}$ on $p(N^{\frac{1}{s}})$ is equivalent to showing that $\delta(N^{\frac{1}{s}}) \cap N = \delta(N)$. Indeed, assume that $\bar{\delta}(p(n_1)) = N\delta(n_1) = N\delta(n_2) = \bar{\delta}(p(n_2))$ for $n_i \in N^{\frac{1}{s}}$, then

$$\delta(n_1)\delta(n_2^{-1}) = \delta(n_1 n_2^{-1}) \in N \cap \delta(N^{\frac{1}{s}}) = \delta(N)$$

and therefore $n_1 n_2^{-1} \in N$ because δ is injective. So this implies $Nn_1 = Nn_2$ and thus that $\bar{\delta}$ is injective on $p(N^{\frac{1}{s}})$.

So the only thing left to show is that $\delta(N^{\frac{1}{s}}) \cap N = \delta(N)$. We know that $\delta(N) \leq \delta(N^{\frac{1}{s}}) \cap N$ and that $\delta(N)$ is a subgroup of index $|D|$ in N. Therefore it suffices to show that also $\delta(N^{\frac{1}{s}}) \cap N$ is a subgroup of index $|D|$ in N to conclude that $\delta(N) = \delta(N^{\frac{1}{s}}) \cap N$. The subgroup $\delta(N^{\frac{1}{s}})$ is a subgroup of index

$|D|$ in $N^{\frac{1}{s}}$. Because of Lemma 6.8 and the fact that $\gcd(s, D) = 1$, we get that $\delta(N^{\frac{1}{s}}) \cap N$ has index $|D|$ in N. Thus $\delta(N^{\frac{1}{s}}) \cap N = \delta(N)$ which finishes the proof. $\square$

The proof of Theorem 7.13 is almost identical to part of the proof of Theorem 6.15, see page 104.

Corollary 7.14. Let $\bar{\alpha}$ be an affine infra-nilmanifold endomorphism of the infra-nilmanifold $\Gamma\backslash G$. Then either $\mathrm{Per}(\bar{\alpha}) = \varnothing$ or $\mathrm{Per}(\bar{\alpha})$ is a dense subset of $\Gamma\backslash G$.

This corollary is exactly Main Theorem 3 from the introduction.

Proof. Assume that $\bar{\alpha}$ has periodic points, then we show that $\mathrm{Per}(\bar{\alpha})$ is dense. Since $\mathrm{Per}(\bar{\alpha}^k) = \mathrm{Per}(\bar{\alpha})$ we can take some power of $\bar{\alpha}$ and assume that $\bar{\alpha}$ has a fixed point. By Theorem 7.7 we can therefore assume that $\bar{\alpha}$ is an infra-nilmanifold endomorphism. An application of Theorem 7.4 and Lemma 7.5 shows that it suffices to consider the case of a nilmanifold endomorphism.

So let $\bar{\delta} : N\backslash G \to N\backslash G$ be a nilmanifold endomorphism. Denote by $p : G \to N\backslash G$ the projection map and let D be the determinant of $\delta \in \mathrm{Aut}(G)$. The set $N_D^{\mathbb{Q}}$ forms a dense subset of G and Lemma 7.5 implies that $p(N_D^{\mathbb{Q}})$ is dense in $\Gamma\backslash G$. Since Theorem 7.13 shows that $p(N_D^{\mathbb{Q}})$ is a subset of $\mathrm{Per}(\bar{\delta})$, this ends the proof. $\square$

Remark 7.15. Let $\bar{\alpha}$ be an affine infra-nilmanifold endomorphism induced by the affine map (g, δ) and assume that Γg_0 is a periodic point. Because of Remark 7.8 and the proof of Corollary 7.14 it follows that $p(g_0 N^{\mathbb{Q}}) \subseteq \mathrm{ePer}(\bar{\alpha})$ and $p(g_0 N_D^{\mathbb{Q}}) \subseteq \mathrm{Per}(\bar{\alpha})$. This gives us a more precise statement of Corollary 7.14.

In this section we constructed a dense subset of periodic points for affine infra-nilmanifold endomorphisms $\bar{\alpha}$. A more general question is to give a full description of the sets $\mathrm{Per}(\bar{\alpha})$ and $\mathrm{ePer}(\bar{\alpha})$. For this we first need a necessary condition for a point Γg of $\Gamma\backslash G$ to be (eventually periodic), which we deduce in the following section.

7.3 Necessary condition for (eventually) periodic points

In this section, we give a necessary condition for a point to be a(n) (eventually) periodic point of a nilmanifold endomorphism. Combined with the results of the previous sections, this will give us a complete description of the set of eventually periodic points, also in the case of general affine infra-nilmanifold endomorphisms.

Let $N^{\mathbb{Q}}$ be the radicable hull of an $\mathcal{F}$-group N with corresponding rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. The radicable subgroups of $N^{\mathbb{Q}}$ correspond to the rational subalgebras of $\mathfrak{n}^{\mathbb{Q}}$. If $\varphi : N^{\mathbb{Q}} \to N^{\mathbb{Q}}$ is a group morphism, then $\ker(\varphi)$ is a radicable normal subgroup of $N^{\mathbb{Q}}$. For every radicable subgroup $H^{\mathbb{Q}} \leq N^{\mathbb{Q}}$, we will denote by $H^{\mathbb{R}}$ the unique simply connected and connected Lie group which contains H as a dense subgroup. For the radicable group $N^{\mathbb{Q}}$ we have $N^{\mathbb{R}} = G$ and thus the groups $H^{\mathbb{R}}$ are Lie subgroups of G. Every automorphism $\varphi : H^{\mathbb{Q}} \to H^{\mathbb{Q}}$ uniquely extends to an element of $\mathrm{Aut}(H^{\mathbb{R}})$ and will denote this extension as $\varphi^{\mathbb{R}}$.

Let G be a simply connected and connected nilpotent Lie group with lattice $N \leq G$ and assume that $\bar{\delta}$ is a nilmanifold endomorphism induced by $\delta \in \mathrm{Aut}(G)$. If the coset Ng is an eventually periodic points of $\bar{\delta}$, then there exist $0 \leq k_1 < k_2$ such that

$$N\delta^{k_1}(g) = N\delta^{k_2}(g)$$

or equivalently such that

$$\delta^{k_2}(g)\left(\delta^{k_1}(g)\right)^{-1} = \delta^{k_2}(g)\delta^{k_1}\left(g^{-1}\right) \in N \leq N^{\mathbb{Q}}.$$

More generally, we are interested in the question which elements n of the real Mal'cev completion $N^{\mathbb{R}}$ of a torsion-free radicable nilpotent group $N^{\mathbb{Q}}$ satisfy a relation of the form

$$\varphi^{\mathbb{R}}(n)\psi^{\mathbb{R}}\left(n^{-1}\right) \in N^{\mathbb{Q}}$$

with $\varphi, \psi : N^{\mathbb{Q}} \to N^{\mathbb{Q}}$ automorphisms of the group $N^{\mathbb{Q}}$. The following theorem gives an answer to this question.

Theorem 7.16. *Let $N^{\mathbb{Q}}$ be a torsion-free radicable nilpotent group with group morphisms $\varphi, \psi : N^{\mathbb{Q}} \to N^{\mathbb{Q}}$ and take $H^{\mathbb{Q}} \leq N^{\mathbb{Q}}$ the radicable subgroup defined as*

$$H^{\mathbb{Q}} = \left\{h \in N^{\mathbb{Q}} \;\middle|\; \varphi(h) = \psi(h)\right\}.$$

Then for all $n \in N^{\mathbb{R}}$ we have the following equivalence:

$$\varphi^{\mathbb{R}}(n)\psi^{\mathbb{R}}(n)^{-1} \in N^{\mathbb{Q}} \iff n \in N^{\mathbb{Q}}H^{\mathbb{R}}.$$

In the statement of Theorem 7.16, $N^{\mathbb{Q}}H^{\mathbb{R}}$ is the product of the subgroups $N^{\mathbb{Q}}$ and $H^{\mathbb{R}}$ which in general does not form a subgroup of $N^{\mathbb{R}}$. Note that the group $H^{\mathbb{Q}}$ is indeed a radicable subgroup, since if $h \in H^{\mathbb{Q}}$, then

$$\varphi(h^{\frac{1}{s}}) = \varphi(h)^{\frac{1}{s}} = \psi(h)^{\frac{1}{s}} = \psi(h^{\frac{1}{s}})$$

and thus $h^{\frac{1}{s}} \in H^{\mathbb{Q}}$. To prove Theorem 7.16, we will use induction depending on the lower central series of $N^{\mathbb{Q}}$, combined with the following easy lemma, which corresponds to the abelian case of Theorem 7.16.

Lemma 7.17. *Let $\varphi : \mathbb{Q}^n \to \mathbb{Q}^m$ be a linear map and take the unique extension $\varphi^{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^m$. Then it holds that $\varphi^{\mathbb{R}}(\mathbb{R}^n) \cap \mathbb{Q}^m = \varphi(\mathbb{Q}^n)$.*

Proof. Consider the rational subspace $V^{\mathbb{Q}} = \ker(\varphi) \subseteq \mathbb{Q}^n$. Take a complementary subspace $W^{\mathbb{Q}} \subseteq \mathbb{Q}^n$ for $V^{\mathbb{Q}}$, meaning that $\mathbb{Q}^n$ can be written as the direct sum $\mathbb{Q}^n = V^{\mathbb{Q}} \oplus W^{\mathbb{Q}}$. The restriction $\varphi|_{W^{\mathbb{Q}}}$ of the linear map φ to the subspace $W^{\mathbb{Q}}$ forms an isomorphism between $W^{\mathbb{Q}}$ and $\varphi(\mathbb{Q}^n)$. Since $\mathbb{R}$ can also be written as the direct sum $\mathbb{R}^n = V^{\mathbb{R}} \oplus W^{\mathbb{R}} = \ker(\varphi^{\mathbb{R}}) \oplus W^{\mathbb{R}}$, $\varphi^{\mathbb{R}}|_{W^{\mathbb{R}}}$ forms an isomorphism between $W^{\mathbb{R}}$ and $\varphi^{\mathbb{R}}(\mathbb{R}^n)$ which is the extension of $\varphi|_{W^{\mathbb{Q}}}$. This implies that $\varphi^{\mathbb{R}}(\mathbb{R}^n) \cap \mathbb{Q}^m = \varphi(\mathbb{Q}^n)$. □

The abelian case, so for linear maps $\varphi, \psi : \mathbb{Q}^n \to \mathbb{Q}^n$, follows from applying Lemma 7.17 to the linear map $\varphi - \psi : \mathbb{Q}^n \to \mathbb{Q}^n$. By using induction, we can now prove the general statement of Proposition 7.16.

Proof of Theorem 7.16. One implication is immediate, namely if $n \in N^{\mathbb{Q}} H^{\mathbb{R}}$, so $n = mh$ for some $m \in N^{\mathbb{Q}}, h \in H^{\mathbb{R}}$, then

$$\begin{aligned}\varphi(n)\psi(n)^{-1} &= \varphi(mh)\psi(mh)^{-1} \\ &= \varphi(m)\varphi(h)\psi(h)^{-1}\psi(m)^{-1} \\ &= \varphi(m)\psi(m)^{-1} \in N^{\mathbb{Q}}.\end{aligned}$$

For the other implication we first introduce some notations. Consider the abelian groups ${}^{\gamma_i(N^{\mathbb{Q}})}\!/\!{}_{\gamma_{i+1}(N^{\mathbb{Q}})}$ with quotient maps $\pi_i : \gamma_i(N^{\mathbb{Q}}) \to {}^{\gamma_i(N^{\mathbb{Q}})}\!/\!{}_{\gamma_{i+1}(N^{\mathbb{Q}})}$. Take $N_1^{\mathbb{Q}}$ equal to the group $N^{\mathbb{Q}}$ and consider $\alpha_1 : N_1^{\mathbb{Q}} \to {}^{N^{\mathbb{Q}}}\!/\!{}_{\gamma_2(N^{\mathbb{Q}})}$ the map defined by

$$\alpha_1(n) = \pi_1\left(\varphi(n)\psi(n)^{-1}\right) = \pi_1(\varphi(n)) - \pi_1(\psi(n)).$$

The map α_1 is a group morphism since for every $m, n \in N_1^{\mathbb{Q}}$, we have

$$\begin{aligned}\alpha_1(mn) &= \pi_1(\varphi(mn)) - \pi_1(\psi(mn)) \\ &= \pi_1(\varphi(m)) - \pi_1(\psi(m)) + \pi_1(\varphi(n)) - \pi_1(\psi(n)) \\ &= \alpha_1(m) + \alpha_1(n).\end{aligned}$$

Denote by $N_2^{\mathbb{Q}}$ the kernel of the group morphism α_1.

Inductively, we also define the group morphism $\alpha_i : N_i^{\mathbb{Q}} \to {}^{\gamma_i(N^{\mathbb{Q}})}\!/_{\gamma_{i+1}(N^{\mathbb{Q}})}$ given by

$$\alpha_i(n) = \pi_i(\varphi(n)\psi(n)^{-1})$$

and the subgroup $N_i^{\mathbb{Q}}$ of $N^{\mathbb{Q}}$ as the kernel of the morphism α_{i-1}. To show that these maps α_i are indeed group morphisms, we first consider the map $\tilde{\alpha}_i : N_i^{\mathbb{Q}} \to {}^{N^{\mathbb{Q}}}\!/_{\gamma_{i+1}(N^{\mathbb{Q}})}$ which is the composition of α_i and the natural inclusion ${}^{\gamma_i(N^{\mathbb{Q}})}\!/_{\gamma_{i+1}(N^{\mathbb{Q}})}$ into ${}^{N^{\mathbb{Q}}}\!/_{\gamma_{i+1}(N^{\mathbb{Q}})}$. A computation shows that

$$\begin{aligned}
\tilde{\alpha}_i(mn) &= \varphi(mn)\psi(mn)^{-1}\gamma_{i+1}(N^{\mathbb{Q}}) \\
&= \varphi(m)\varphi(n)\psi(n)^{-1}\psi(m)^{-1}\gamma_{i+1}(N^{\mathbb{Q}}) \\
&= \varphi(m)\tilde{\alpha}_i(n)\psi(m)^{-1}\gamma_{i+1}(N^{\mathbb{Q}}) \\
&= \tilde{\alpha}_i(m)\tilde{\alpha}_i(n)[\tilde{\alpha}_i(n), \psi(m)^{-1}]\gamma_{i+1}(N^{\mathbb{Q}}) \\
&= \tilde{\alpha}_i(m)\tilde{\alpha}_i(n)
\end{aligned}$$

where the last equality holds because $\tilde{\alpha}_i(n) \in {}^{\gamma_i(N^{\mathbb{Q}})}\!/_{\gamma_{i+1}(N^{\mathbb{Q}})}$. This shows that the map $\tilde{\alpha}_i$ is a group morphism and therefore also the map α_i.

The groups $N_i^{\mathbb{Q}}$ are radicable subgroups of $N^{\mathbb{Q}}$ and thus we can consider the groups $N_i^{\mathbb{R}}$. The group $H^{\mathbb{Q}}$ is a subgroup of every group $N_i^{\mathbb{Q}}$ since for every $h \in H^{\mathbb{Q}}$ we have $\varphi(h)\psi(h)^{-1} = 0$. Every group $N_{i+1}^{\mathbb{Q}}$ is a normal subgroup of $N_i^{\mathbb{Q}}$ and we have group morphisms

$$\bar{\alpha}_i : {}^{N_i^{\mathbb{Q}}}\!/_{N_{i+1}^{\mathbb{Q}}} \to {}^{\gamma_i(N^{\mathbb{Q}})}\!/_{\gamma_{i+1}(N^{\mathbb{Q}})}.$$

The extension $\bar{\alpha}_i^{\mathbb{R}}$ is given by

$$\bar{\alpha}_i^{\mathbb{R}} : {}^{N_i^{\mathbb{R}}}\!/_{N_{i+1}^{\mathbb{R}}} \to {}^{\gamma_i(N^{\mathbb{R}})}\!/_{\gamma_{i+1}(N^{\mathbb{R}})}$$

with $\bar{\alpha}_i^{\mathbb{R}}(n) = \pi_i^{\mathbb{R}}(\varphi^{\mathbb{R}}(n)\psi^{\mathbb{R}}(n)^{-1})$. Since $N^{\mathbb{Q}}$ is c-step nilpotent, the group ${}^{\gamma_i(N^{\mathbb{Q}})}\!/_{\gamma_{i+1}(N^{\mathbb{Q}})} = 1$ for every $i \geq c+1$ and therefore the groups $N_i^{\mathbb{Q}} \leq H^{\mathbb{Q}}$ for every $i \geq c+1$. In particular we get $H = N_{c+1}^{\mathbb{Q}}$.

With these groups $N_i^{\mathbb{Q}}$, we can prove the other implication by using induction on the subgroups $N_i^{\mathbb{R}}$. If $n \in N_{c+1}^{\mathbb{R}}$, there is nothing to prove since $N_{c+1}^{\mathbb{R}} = H^{\mathbb{R}}$.

Now assume that the proposition is true for every $n \in N^{\mathbb{R}}_{i+1}$, then we will show it is true for $n \in N^{\mathbb{R}}_i$. Consider the map $\bar{\alpha}_i : {N^{\mathbb{Q}}_i}/{N^{\mathbb{Q}}_{i+1}} \to {\gamma_i(N^{\mathbb{Q}})}/{\gamma_{i+1}(N^{\mathbb{Q}})}$ with extension $\bar{\alpha}^{\mathbb{R}}_i : {N^{\mathbb{R}}_i}/{N^{\mathbb{R}}_{i+1}} \to {\gamma_i(N^{\mathbb{R}})}/{\gamma_{i+1}(N^{\mathbb{R}})}$. This map satisfies the properties of Lemma 7.17, so there exists $m \in N^{\mathbb{Q}}_i$ such that $\bar{\alpha}^{\mathbb{R}}_i(m) = \bar{\alpha}^{\mathbb{R}}_i(n)$ or thus that $\tilde{n} = m^{-1}n \in N^{\mathbb{R}}_{i+1}$. Since $m \in N^{\mathbb{Q}}$ and thus also $\varphi(m^{-1}), \psi(m) \in N^{\mathbb{Q}}$, we have that

$$\varphi(\tilde{n})\psi(\tilde{n}^{-1}) = \varphi(m^{-1}) \underbrace{\varphi(n)\psi(n^{-1})}_{\in N^{\mathbb{Q}}} \psi(m) \in N^{\mathbb{Q}}.$$

By induction we conclude that $n \in N^{\mathbb{Q}} H^{\mathbb{R}}$. □

As discussed above Theorem 7.16, this results makes it possible to study (eventually) periodic points. A combination with Theorem 7.9 gives us the following result.

Theorem 7.18. *Let $\bar{\delta} : N\backslash G \to N\backslash G$ be a nilmanifold endomorphism induced by the map $\delta \in \operatorname{Aut}(G)$ on the nilmanifold $N\backslash G$ with projection map $p : G \to N\backslash G$. Let $H^{\mathbb{R}}$ be the subgroup of G defined as*

$$H^{\mathbb{R}} = \left\{ h \in G \;\middle|\; \exists k > 0 : \delta^k(h) = h \right\},$$

then

$$\operatorname{ePer}(\bar{\delta}) = p(N^{\mathbb{Q}} H^{\mathbb{R}}).$$

The subgroup $H^{\mathbb{R}}$ of G of this theorem is exactly the subgroup corresponding to the eigenspaces of δ corresponding to roots of unity. This subgroup can thus be easily computed starting from the automorphism δ.

Proof. From the discussion above Theorem 7.16, we know that an eventually periodic point $p(g) = Ng$ has to satisfy $\delta^{k_2}(g)\delta^{k_1}\left(g^{-1}\right) \in N^{\mathbb{Q}}$ for some $0 < k_1 < k_2$. Since $\delta^{k_1} : N^{\mathbb{Q}} \to N^{\mathbb{Q}}$ is invertible, this is equivalent to $\delta^k(g)g^{-1} \in N^{\mathbb{Q}}$ for some $k > 0$. Theorem 7.16 applied to δ^k and $\mathbb{1}_{N^{\mathbb{Q}}}$ implies that g is indeed an element of $N^{\mathbb{Q}} H^{\mathbb{R}}$, showing that the inclusion $\operatorname{ePer}(\bar{\delta}) \subseteq p(N^{\mathbb{Q}} H^{\mathbb{R}})$ indeed holds.

Thus it suffices to show that every point of $p(N^{\mathbb{Q}} H^{\mathbb{R}})$ is eventually periodic. By replacing δ by a power of itself, we can assume that $h \in H^{\mathbb{R}}$ implies $\delta(h) = h$. Let $g = nh$ with $n \in N^{\mathbb{Q}}$ and $h \in H^{\mathbb{R}}$, then Theorem 7.9 implies that $p(n)$ is eventually periodic or thus that there exist distinct $k_1, k_2 > 0$ such that

$$\bar{\delta}^{k_1}(p(n)) = \bar{\delta}^{k_2}(p(n)).$$

Now

$$\bar{\delta}^{k_1}(p(g)) = N\delta^{k_1}(n)\delta^{k_1}(h) = N\delta^{k_2}(n)h = N\delta^{k_2}(n)\delta^{k_2}(h) = \bar{\delta}^{k_2}(p(g))$$

and thus $p(g)$ is eventually periodic. □

By combining Theorem 7.18 with Theorem 7.7 (and the explicit form of the homeomorphism as described in Remark 7.8), we get the following corollary for affine infra-nilmanifold endomorphisms. This is exactly Main Theorem 4 from the introduction.

Corollary 7.19. Let $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ be an affine infra-nilmanifold endomorphism induced by the affine transformation $\alpha = (g, \delta) \in \mathrm{Aff}(G)$. Denote by $p : G \to \Gamma\backslash G$ the projection map and by $H^{\mathbb{R}}$ the subgroup of G defined as

$$H^{\mathbb{R}} = \{h \in G \mid \exists k > 0 : \delta^k(h) = h\}.$$

If $\bar{\alpha}$ has a periodic point Γg_0, then

$$\mathrm{ePer}(\bar{\alpha}) = p(N^{\mathbb{Q}} g_0 H^{\mathbb{R}}).$$

Note that $N^{\mathbb{Q}}$ in the statement is the radicable hull of the Fitting subgroup of Γ, which is in general different from the radicable hull of the infra-nilmanifold endomorphism $\bar{\delta}$ constructed by Theorem 7.7.

For periodic points there is a similar description. The only problem is that we do not know which points of $p(N^{\mathbb{Q}})$ are periodic, since we only constructed a subset in Theorem 7.13.

Theorem 7.20. *Let $\bar{\delta} : N\backslash G \to N\backslash G$ be a nilmanifold endomorphism induced by the map $\delta \in \mathrm{Aut}(G)$ on the nilmanifold $N\backslash G$ with projection map $p : G \to N\backslash G$. Let $H^{\mathbb{R}}$ be the subgroup of G defined as*

$$H^{\mathbb{R}} = \{h \in G \mid \exists k > 0 : \delta^k(h) = h\}$$

and take $X = p(N^{\mathbb{Q}}) \cap \mathrm{Per}(\bar{\delta})$. Then

$$\mathrm{Per}(\bar{\delta}) = p(p^{-1}(X)\, H^{\mathbb{R}}).$$

Proof. By taking some power of δ, we can assume that $\delta(h) = h$ for all $h \in H^{\mathbb{R}}$. Let $p(g)$ be a periodic point of $\bar{\delta}$. Since every periodic point is eventually periodic, we can assume that g is of the form $g = nh$ with $n \in N^{\mathbb{Q}}$ and $h \in H^{\mathbb{R}}$ by Theorem 7.18. If $p(g)$ is periodic, we know that $\delta^k(g)g^{-1} \in N$ for some $k > 0$. Since

$$\delta^k(g)g^{-1} = \delta^k(n)\delta^k(h)h^{-1}n^{-1} = \delta^k(n)n^{-1} \in N$$

this implies that $p(n)$ is a periodic point, so $p(n) \in X$. We conclude that $\mathrm{Per}(\bar{\delta}) \subseteq p(p^{-1}(X) H^{\mathbb{R}})$.

The other inclusion is identical as in Theorem 7.18. $\square$

Again, a combination of Theorem 7.20 and Theorem 7.7 gives us the following result.

Corollary 7.21. Let $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ be an affine infra-nilmanifold endomorphism induced by the affine transformation $\alpha = (g, \delta) \in \mathrm{Aff}(G)$ and assume that $\bar{\alpha}$ has a periodic point Γg_0. Denote by $p : G \to \Gamma\backslash G$ the projection map and by $H^{\mathbb{R}}$ the subgroup of G defined as

$$H^{\mathbb{R}} = \{h \in G \mid \exists k > 0 : \delta^k(h) = h\}$$

and take $X = p(N^{\mathbb{Q}}) \cap \mathrm{Per}(\bar{\delta})$ where $\bar{\delta}$ is the nilmanifold endomorphism induced by δ on $g_0^{-1} N g_0$. The periodic points of $\bar{\alpha}$ are equal to

$$\mathrm{Per}(\bar{\alpha}) = p(p^{-1}(X) g_0 H^{\mathbb{R}}).$$

Theorem 7.20 shows us that for a full description of $\mathrm{Per}(\bar{\alpha})$, we only need to know how the set $X = \mathrm{Per}(\bar{\delta}) \cap p(N^{\mathbb{Q}})$ looks like. In Section 7.4 we give some examples showing that this set can be quite wild. The only information we have for general nilmanifold endomorphisms up till now is that $p(X_D) \subseteq X$.

For infra-nilmanifold automorphisms, which have determinant ± 1, we know that $\mathrm{Per}(\bar{\delta}) \supseteq p(N_1^{\mathbb{Q}}) = p(N^{\mathbb{Q}})$ by Theorem 7.13. This implies that for every affine infra-nilmanifold automorphism $\bar{\alpha}$ we have that $\mathrm{Per}(\bar{\alpha}) = \mathrm{ePer}(\bar{\alpha})$ by Theorem 7.20. This behavior completely describes these affine infra-nimanifold automorphisms.

Theorem 7.22. *Let $\bar{\alpha}$ be an affine infra-nilmanifold endomorphism. The map $\bar{\alpha}$ is an affine infra-nilmanifold automorphism (so $\bar{\alpha}$ is a diffeomorphism) if and only if* $\mathrm{Per}(\bar{\alpha}) = \mathrm{ePer}(\bar{\alpha}) \neq \varnothing$.

Proof. It suffices to prove this in the case of a nilmanifold endomorphism $\bar{\delta}$ by Theorem 7.4. One direction was shown just above the theorem. For the other direction, assume that $\bar{\delta}$ is not an nilmanifold automorphism, so its determinant D satisfies $|D| > 1$. Let $N\backslash G$ be the nilmanifold and consider the uniform lattice $N_0 = \delta^{-1}(N) \leq G$ of G. Then N is a finite index subgroup of N_0 of index $|D|$ by Proposition 2.30.

Consider the set $p(N_0)$ which is a set of order $|D|$. After applying the map $\bar{\delta}$ we get that

$$\bar{\delta}\left(p\left(N_0\right)\right) = p\left(\delta\left(N_0\right)\right) = p(N)$$

and hence the only periodic point in $p(\tilde{N})$ is Ne. Since $|D| > 1$, this implies that there exists an eventually periodic point which is not periodic. $\square$

7.4 Examples

In this section we compute the set of periodic points for some concrete examples of affine infra-nilmanifold endomorphisms. These examples shows that in general it is a hard task to describe the set of periodic points, contrary to the eventually periodic points which are completely described in Theorem 7.18.

For abelian groups, the relative order of elements in $N^{\mathbb{Q}}$ also helps to show that points are not periodic.

Proposition 7.23. *Let $\mathbb{T}^n = \mathbb{Z}^n \backslash \mathbb{R}^n$ be an n-torus with projection map $p : \mathbb{R}^n \to \mathbb{Z}^n$ and let f_A be a toral endomorphism induced by the matrix $A \in \mathrm{GL}(n, \mathbb{Q})$. Consider a periodic point $p(q)$ with $q \in \mathbb{Q}^n$, then*

$$\mathrm{ord}_{\mathbb{Z}^n}\left(A^k(q)\right) = \mathrm{ord}_{\mathbb{Z}^n}(q)$$

for all $k > 0$.

Proof. Note that for abelian groups the set

$$X_s = \{q \in \mathbb{Q}^n \mid \ \mathrm{ord}_{\mathbb{Z}^n}(q) \mid s\} \subseteq \mathbb{Q}^n$$

does form a subgroup of $\mathbb{Q}^n$. Because $\mathbb{Z}^n \leq X_s$, we have that $p(q) \in p(X_s)$ if and only if $q \in X_s$.

Assume that $s = \mathrm{ord}_{\mathbb{Z}^n}\left(A^k(q)\right) \leq \mathrm{ord}_{\mathbb{Z}^n}(q)$ for some $q \in \mathbb{Q}^n$. Then $p(q) \notin p(X_s)$ but $f_A^k\left(p\left(q\right)\right) \in p(X_s)$ and this last set is invariant under f_A. This implies that $p(q)$ is not periodic. $\square$

For general nilpotent groups N, there is the following weaker version of this proposition.

Proposition 7.24. *Let $\bar{\delta}$ be a nilmanifold endomorphism on the nilmanifold $N \backslash G$ induced by $\delta \in \mathrm{Aut}(G)$. Denote by $p : G \to N \backslash G$ the natural projection map and let $p(n)$ be a periodic point for $n \in N^{\mathbb{Q}}$. If q is a prime such that $q \mid \mathrm{ord}_N(n)$, then $q \mid \mathrm{ord}_N(\delta^k(n))$ for all $k > 0$.*

Proof. Assume that there exists a prime q and a $k > 0$ such that $q \mid \mathrm{ord}_N(n)$, but $q \nmid \mathrm{ord}_N(\delta^k(n))$. Write $s = \mathrm{ord}_N(\delta^k(n))$, then we have that $n \notin N^{\frac{1}{s}}$ by

Proposition 7.11. Hence $p(n) \notin p(N^{\frac{1}{s}})$ since $N^{\frac{1}{s}}$ is a subgroup of $N^{\mathbb{Q}}$. The point $\bar{\delta}^k(p(n)) \in p(N^{\frac{1}{s}})$ and because $\bar{\delta}$ maps the set $p(N^{\frac{1}{s}})$ to itself, the point $p(n)$ cannot be periodic. $\square$

Both Propositions 7.23 and 7.24 are useful for showing that certain points are not periodic. We will apply these results in the following examples.

In Theorem 7.13, we showed that $p(N_D^{\mathbb{Q}})$ is a subset of the periodic points for an infra-nilmanifold endomorphism of determinant D. In some cases, the set of periodic points will be equal to the subset $p(N_D^{\mathbb{Q}})$.

Example 7.25. Let $\mathfrak{n}^{\mathbb{Q}}$ be a nilpotent Lie algebra which has a positive grading and denote by $N^{\mathbb{Q}}$ the corresponding radicable nilpotent group. Consider the full subgroup N and group morphisms ψ_p as constructed in Corollary 6.3 with $\det(\psi_p)$ some power of p.

For every $n \in N^{\mathbb{Q}}$ with $p \mid \mathrm{ord}_N(n)$, we have that $\mathrm{ord}_N(\delta(n)) \mid \frac{\mathrm{ord}_N(n)}{p}$. So for some power δ^k we have that $\gcd(\mathrm{ord}_N(\delta^k(n)), p) = 1$. This implies that $\mathrm{Per}(\bar{\psi}_p) = p(N_p^{\mathbb{Q}})$ and thus equality holds in Theorem 7.13. In this case, $\mathrm{ePer}(\bar{\psi}_p) \setminus \mathrm{Per}(\bar{\psi}_p)$ is a dense subset of the nilmanifold.

Similarly there is a closed formula for the periodic points of toral endomorphisms induced by diagonal matrices on $\mathbb{T}^n$ by using Proposition 7.23 in the same way. In the case of nilmanifold endomorphisms which are not induced by a diagonal matrix, the situation gets a lot more complicated.

Example 7.26. Consider the 2-torus $\mathbb{T}^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2$ with natural projection map $p : \mathbb{R}^2 \to \mathbb{T}^2$. Denote the nilmanifold endomorphism induced by a matrix $A \in \mathrm{GL}(2, \mathbb{Q})$ as $f_A : \mathbb{T}^2 \to \mathbb{T}^2$. We will compute $\mathrm{Per}(f_A)$ for the matrices

$$A_1 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} 5 & 2 \\ -1 & 1 \end{pmatrix}.$$

Since for every $k > 0$, $A_i^k - I$ is invertible over $\mathbb{Q}$, we know that $\mathrm{ePer}(f_{A_i}) = p(\mathbb{Q}^2)$ in all four cases. This implies that the periodic points also form a subset of $p(\mathbb{Q}^2)$. Note that the first two matrices are hyperbolic (so the maps f_{A_i} are Anosov endomorphisms in these cases) whereas the last two matrices induce expanding maps.

$\boxed{A_1}$ For the first example, we have

$$(A_1)^2 = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} = 2 \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

with $\det\left(\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}\right) = 1$. Therefore, if $q \in \mathbb{Q}^2$ satisfies $2 \mid \operatorname{ord}_{\mathbb{Z}^2}(q)$ then $\operatorname{ord}_{\mathbb{Z}^2}\left((A_1)^2(q)\right) = \frac{\operatorname{ord}_{\mathbb{Z}^2} q}{2}$ and thus $p(q)$ is not periodic by Proposition 7.23. By Theorem 7.13 we conclude that

$$\operatorname{Per}(f_{A_1}) = p(N_2^{\mathbb{Q}}).$$

$\boxed{A_2}$ In the second case, we claim that $\operatorname{ePer}(f_{A_2}) \setminus \operatorname{Per}(f_{A_2}) \supseteq p(X)$ with $X \subseteq \mathbb{Q}^2$ given by

$$X = \left\{\left(\frac{q_1}{2^k}, \frac{q_2}{2^l}\right) \mid q_1, q_2 \in \mathbb{Q},\ k > l \geq 0,\ \gcd(\operatorname{ord}_{\mathbb{Z}}(q_i), 2) = 1\right\}.$$

Note that the set X^c is invariant under A_2.

To see that $p(X) \cap \operatorname{Per}(f_{A_2}) = \varnothing$, write

$$A_2 = \begin{pmatrix}4 & 1\\ 2 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$$

where $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ has determinant 1. If $x \in X$, then $\operatorname{ord}_{\mathbb{Z}^2}(A_2(x)) = \frac{\operatorname{ord}_{\mathbb{Z}^2}(x)}{2}$, so $p(x)$ is indeed not periodic by Proposition 7.23.

On the other hand, if $x = (q_1, q_2) \notin X$ with $2^k \mid \operatorname{ord}_{\mathbb{Z}^2}(x)$, then $2^k \mid \operatorname{ord}_{\mathbb{Z}}(q_2)$ and thus also $2^k \mid \operatorname{ord}_{\mathbb{Z}}(2q_1 + q_2)$ which is the second component of $A_2(x)$. We conclude that if $x \notin X$, then $\operatorname{ord}_{\mathbb{Z}^2}(x) = \operatorname{ord}_{\mathbb{Z}^2}(A_2(x))$. So the points of relative order exactly s in X^c project to a (finite) set which is invariant under f_{A_2}. This implies by Lemma 7.1 that for every $s \in \mathbb{N}$, f_{A_2} has a periodic point $p(x)$ with $\operatorname{ord}_{\mathbb{Z}^2}(x) = s$. So $\operatorname{Per}(f_{A_2}) \supsetneq p(N_2^{\mathbb{Q}})$.

$\boxed{A_3}$ For the third matrix, we have that

$$(A_3)^2 = \begin{pmatrix}-2 & 6\\ -2 & -2\end{pmatrix} = 2\begin{pmatrix}-1 & 3\\ -1 & -1\end{pmatrix}$$

and thus similarly as in the case A_1, we conclude that $\operatorname{Per}(f_{A_3}) = p(N_2^{\mathbb{Q}})$.

$\boxed{A_4}$ In the last example, we find that $\operatorname{ePer}(f_{A_4}) \setminus \operatorname{Per}(f_{A_4}) \supseteq p(Y)$ with

$$Y = \left\{\left(\frac{q_1}{7^k}, \frac{q_2}{7^k}\right) \middle|\ k > 0,\ \gcd(\operatorname{ord}_{\mathbb{Z}}(q_i), 7) = 1 = \gcd\left(\operatorname{ord}_{\mathbb{Z}}\left(\frac{q_1 - q_2}{7}\right), 7\right)\right\}.$$

Also for every $s \in \mathbb{N}$, we have a periodic points $p(x)$ such that $\operatorname{ord}_{\mathbb{Z}^2}(x) = s$. The proof is similar as in the case A_2.

So these examples show that Theorem 7.13 sometimes but not always gives us the set of periodic points, even not in the special case of Anosov endomorphisms or expanding maps. Note that in all examples above, the set of eventually periodic points which are not periodic form a dense subset of the infra-nilmanifold. In Chapter 11 we formulate a conjecture concerning the affine infra-nilmanifold endomorphisms having this property.

7.5 Generalization to other types of self-maps

In this section, we discuss the generalization of some results in this chapter to a more general class of self-maps on infra-nilmanifolds, namely the ones induced by elements in $\mathrm{aff}(G)$. Since the linear part of such a map is not always invertible, there are some difficulties in generalizing the previous results, certainly the results about periodic points where we use the injective version of Lemma 7.1.

Reduction to nilmanifolds

Let $\Gamma\backslash G$ is an infra-nilmanifold and $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ a map induced by $\alpha \in \mathrm{aff}(G)$. Contrary to the situation of affine infra-nilmanifold endomorphisms, $\bar{\alpha}$ does not always have a lift to the nilmanifold $N\backslash G$ with $N = \Gamma \cap G$. There does exist a nilmanifold though which finitely covers $\Gamma\backslash G$ and such that every map has a lift to this nilmanifold.

Theorem 7.27. *Let $\Gamma\backslash G$ be an infra-nilmanifold, then there exists a nilmanifold $N_0\backslash G$ such that every map $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ induced by an $\alpha \in \mathrm{aff}(G)$ has a lift $\tilde{\alpha}$ to $N_0\backslash G$. Moreover, if $p : N_0\backslash G \to \Gamma\backslash G$ is the covering map, then*

$$p^{-1}\left(\mathrm{ePer}(\bar{\alpha})\right) = \mathrm{ePer}(\tilde{\alpha})$$

$$\mathrm{Per}(\bar{\alpha}) = p\left(\mathrm{Per}(\tilde{\alpha})\right).$$

For giving a description of the set $\mathrm{Per}(\bar{\alpha})$ and studying its density, this weaker version of Theorem 7.4 is enough.

Proof. Let F be the holonomy group of Γ and take

$$N_0 = \Gamma^{|F|} = \langle \gamma^{|F|} \mid \gamma \in \Gamma \rangle \leq G.$$

Since N_0 is a fully characteristic subgroup, every map has a lift to $N_0\backslash G$. The other claims follow immediately from Proposition 7.2. $\square$

The other results of Section 7.1 generalize without problem. In particular, also Theorem 7.7 is true for maps induced by affine maps of $\mathrm{aff}(G)$.

Theorem 7.28. *Let $\bar{\alpha}$ be a map on the infra-nilmanifold $\Gamma\backslash G$, then $\bar{\alpha}$ is topologically conjugate to a map $\bar{\delta}$ induced by $\delta \in \mathrm{Endo}(G)$ on some infra-nilmanifold if and only if it has a fixed point.*

Sufficient condition

The results of this paragraph about eventually periodic points generalize immediately.

Theorem 7.29. *Let N be a lattice of the nilpotent Lie group G with radicable hull $N^{\mathbb{Q}}$. If $\bar{\delta} : N\backslash G \to N\backslash G$ is a map induced by a group morphism $G \to G$ then*

$$p(N^{\mathbb{Q}}) \subseteq \mathrm{ePer}(\bar{\delta}).$$

The results about periodic points are harder to generalize and therefore are not discussed here. It is certainly not true that the set of periodic points of such a map is dense in the manifold.

Necessary condition

Since Theorem 7.16 is true for general group morphisms of $\mathcal{F}$-groups, we can still use this result in the generalized case.

Theorem 7.30. *Let $\Gamma\backslash G$ be an infra-nilmanifold and $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$ a map induced by the affine map $\alpha = (g, \delta) \in \mathrm{aff}(G)$. If $\bar{\alpha}$ has a periodic point Γg_0, then*

$$\mathrm{ePer}(\bar{\alpha}) = p(g_0 N^{\mathbb{Q}} H^{\mathbb{R}})$$

with $H^{\mathbb{R}} = \{h \in G \mid \exists k_1, k_2 > 0 : \delta^{k_1}(h) = \delta^{k_2}(h)\}$

The subgroup $H^{\mathbb{R}}$ can be described as the generalized eigenspace corresponding to the eigenvalue 0 and the eigenspaces of eigenvalues which are roots of unity.

Proof. The proof is identical to the proof of Theorem 7.18, except for the step where we use that $\delta^{k_1} : N^{\mathbb{Q}} \to N^{\mathbb{Q}}$ is invertible. Instead, we use Theorem 7.16 directly on the maps δ^{k_1} and δ^{k_2} to get the result. $\square$

So the situation for eventually periodic points can be completely generalized to maps induced by $\mathrm{aff}(G)$. How the set of periodic points for these maps looks like is still completely open.

Chapter 8

Gradings on Lie algebras

In this chapter we discuss the existence of certain gradings on nilpotent Lie algebras. As we showed in Chapter 6, we are interested in two types of gradings, namely positive gradings on the one hand and non-trivial non-negative gradings on the other hand. The main method of this chapter is to link the existence of these gradings to the existence of certain automorphisms of the Lie algebra.

In Section 8.2 we state the exact nature of this relation between automorphisms and gradings on a Lie algebra. The main consequence is that the techniques of linear algebraic groups can be used to study gradings on Lie algebras. In the following sections, we then use the structure of linear algebraic groups to show that the existence of such a grading is invariant under field extensions and that we can always adapt the grading such that it is invariant under a given finite group of automorphisms. The most important application is on infra-nilmanifolds and leads to Main Theorem 1 and 2, but we show how these results are also important for studying Lie algebras on their own.

Some parts of this chapter also fit in other chapters, as was done in the original papers [32, 42], but we have chosen to group all arguments concerning gradings on Lie algebras in this chapter. All the fields we consider are subfields of the complex numbers. This is necessary to define the notions of expanding and partially expanding elements of linear algebraic groups.

8.1 Background and definitions

This first section gives some background about gradings of Lie algebras, which is an interesting topic on its own. During this section, $\mathfrak{n}^E$ is a finite dimensional Lie algebra over the field E.

Definition 8.1. A **grading** on $\mathfrak{n}^E$ is a decomposition of $\mathfrak{n}^E$ as a direct sum

$$\mathfrak{n}^E = \bigoplus_{i \in \mathbb{Z}} \mathfrak{n}^E_i$$

of subspaces $\mathfrak{n}^E_i \subseteq \mathfrak{n}^E$ such that

$$[\mathfrak{n}^E_i, \mathfrak{n}^E_j] \subseteq \mathfrak{n}^E_{i+j}$$

for all $i, j \in \mathbb{Z}$. An automorphism $\varphi \in \operatorname{Aut}(\mathfrak{n}^E)$ preserves this grading if $\varphi(\mathfrak{n}^E_i) = \mathfrak{n}^E_i$ for every i. We call the grading **positive** if $\mathfrak{n}^E_i = 0$ for every $i \leq 0$. The grading is **non-negative** if $\mathfrak{n}^E_i = 0$ for every $i < 0$.

There are many examples of gradings on Lie algebras.

Example 8.2.

(i) Every Lie algebra $\mathfrak{n}^E$ has a trivial grading given by $\mathfrak{n}^E = \mathfrak{n}^E_0$.

(ii) Let $\mathfrak{n}^E$ be an abelian Lie algebra, meaning that the bracket of $\mathfrak{n}^E$ is trivial. Every decomposition of $\mathfrak{n}^E$ into a direct sum of vector spaces $\mathfrak{n}^E_i$, so

$$\mathfrak{n}^E = \bigoplus_{i \in \mathbb{Z}} \mathfrak{n}^E_i,$$

is a grading on the Lie algebra $\mathfrak{n}^E$. The condition $[\mathfrak{n}^E_i, \mathfrak{n}^E_j] \subseteq \mathfrak{n}^E_{i+j}$ is trivially satisfied in this case since $[\mathfrak{n}^E_i, \mathfrak{n}^E_j] = 0$ for every $i, j \in \mathbb{Z}$. In particular, the abelian Lie algebra $\mathfrak{n}^E$ has a positive grading, for example by taking $\mathfrak{n}^E = \mathfrak{n}^E_1$.

(iii) As explained in Proposition 6.25, every 2-step nilpotent Lie algebra $\mathfrak{n}^E$ has a positive grading with $\mathfrak{n}^E_2 = \gamma_2(\mathfrak{n}^E)$.

(iv) Let $\mathfrak{n}^E$ be a characteristically nilpotent Lie algebra, then the only grading on $\mathfrak{n}^E$ is the trivial grading. Indeed, if

$$\mathfrak{n}^E = \bigoplus_{i \in \mathbb{Z}} \mathfrak{n}^E_i$$

is a grading, then consider the automorphism $\varphi : \mathfrak{n}^E \to \mathfrak{n}^E$ defined by

$$X \mapsto 2^i X \ \ \forall X \in \mathfrak{n}_i^E.$$

From the definition of grading, it follows that this is indeed an automorphism. Since $\mathfrak{n}^E$ has only unipotent automorphisms, this implies that $\mathfrak{n}^E = \mathfrak{n}_0^E$.

(v) The subspace $\mathfrak{sl}(2, E)$ of matrices with trace 0 forms a subalgebra of the matrix Lie algebra $E^{2\times 2}$. This Lie algebra has a basis X, H, Y given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A computation shows that this basis satisfies the relations

$$[X, Y] = H, \ [H, X] = 2X, \ [H, Y] = -2Y.$$

So the decomposition

$$\mathfrak{sl}(2, E) = \mathfrak{n}_{-1}^E \oplus \mathfrak{n}_0^E \oplus \mathfrak{n}_1^E$$

given by $\mathfrak{n}_1^E = \langle X \rangle$, $\mathfrak{n}_0^E = \langle H \rangle$, $\mathfrak{n}_{-1}^E = \langle Y \rangle$ is a grading on the Lie algebra $\mathfrak{sl}(2, E)$. This Lie algebra is not nilpotent since it follows that $\gamma_2(\mathfrak{sl}(2, E)) = \mathfrak{sl}(2, E)$ from the relations above.

So gradings exist on many Lie algebras, but in the remaining part of this chapter we focus on nilpotent Lie algebras, motivated by the following result.

Theorem 8.3. *Let $\mathfrak{n}^E$ be a Lie algebra which has a grading $\mathfrak{n}^E = \bigoplus_{i\in\mathbb{Z}} \mathfrak{n}_i^E$ with $\mathfrak{n}_0^E = 0$, then the Lie algebra $\mathfrak{n}^E$ is nilpotent.*

Theorem 8.3 is a consequence of the main result in [69], which is the following.

Theorem 8.4. *Let $\mathfrak{n}^E$ be a Lie algebra and* $\mathrm{Der}(\mathfrak{n}^E)$ *the Lie algebra of derivations of $\mathfrak{n}^E$. Let D be a subalgebra of* $\mathrm{Der}(\mathfrak{n}^E)$ *such that*

(1) *the subalgebra D is nilpotent;*

(2) *if $X \in \mathfrak{n}^E$ such that $d(X) = 0$ for every $d \in D$, then $X = 0$.*

Then the Lie algebra $\mathfrak{n}^E$ is nilpotent.

Since it is not clear on first sight how these results are related, we show how the proof of Theorem 8.3 follows from Theorem 8.4.

Proof of Theorem 8.3. Given a grading on the Lie algebra $\mathfrak{n}^E = \bigoplus_{i\in\mathbb{Z}} \mathfrak{n}_i^E$, we define a derivation $d : \mathfrak{n}^E \to \mathfrak{n}^E$ by

$$d(X) = iX$$

for every $X \in \mathfrak{n}_i^E$. To see that d is really a derivation, it suffices to check the definition for a basis of $\mathfrak{n}^E$. Note that for every $X \in \mathfrak{n}_i^E$ and $Y \in \mathfrak{n}_j^E$, we have that

$$d([X,Y]) = (i+j)[X,Y] = i[X,Y] + j[X,Y] = [d(X),Y] + [X,d(Y)],$$

and so d is a derivation because there is a basis where each basis vector lies in some subspace $\mathfrak{n}_i^E$.

Now take D the 1-dimensional vector space spanned by the derivation d. Since $[d,d] = 0$, this is a nilpotent subalgebra of $\mathrm{Der}(\mathfrak{n}^E)$. The kernel of d is given by the vectors $X \in \mathfrak{n}^E$ such that $d(X) = 0$ and this is exactly the subspace $\mathfrak{n}_0^E$. So if $\mathfrak{n}_0^E = 0$, this implies that the Lie algebra is nilpotent by Theorem 8.4. □

In particular, Theorem 8.3 implies that every Lie algebra with a positive grading is necessarily nilpotent. Nilpotent Lie algebras are closely related to infra-nilmanifolds as introduced in Chapter 2.

Sometimes a grading is defined as a decomposition

$$\mathfrak{n}^E = \bigoplus_{r\in\mathbb{R}} \mathfrak{n}_r^E$$

which is indexed by real numbers rather than integers as in our definition. We call a grading of this form an $\mathbb{R}$-grading for the Lie algebra $\mathfrak{n}^E$. These definitions are equivalent because of the following proposition, which was first given in [34, Proposition 2.6.].

Proposition 8.5. *Let $\mathfrak{n}^E$ be a Lie algebra with $\mathbb{R}$-grading*

$$\mathfrak{n}^E = \bigoplus_{r\in\mathbb{R}} \mathfrak{n}_r^E.$$

By renaming the vector spaces $\mathfrak{n}_r^E$, there also exists a grading $\mathfrak{n}^E = \bigoplus_{i\in\mathbb{Z}} \mathfrak{n}_i^E$ for $\mathfrak{n}^E$. If the original $\mathbb{R}$-grading was positive or non-negative and non-trivial, we can find such a grading which is also positive or non-negative and non-trivial.

Proof. Since $\mathfrak{n}^E$ is a finite dimensional Lie algebra, only a finite number of the subspaces $\mathfrak{n}_r^E$ are non-trivial, i.e. different from 0. Let $r_1, \dots, r_n$ be the real numbers with $\mathfrak{n}_{r_i}^E \neq 0$ and consider the vector $(r_1, \dots, r_n)$ of $\mathbb{R}^n$. There are also only a finite number of r_i, r_j, r_k with $0 \neq [\mathfrak{n}_{r_i}^E, \mathfrak{n}_{r_j}^E] \subseteq \mathfrak{n}_{r_k}^E$.

Let V be the subspace of $\mathbb{R}^n$ determined by these equations, so $v = (v_1, \ldots, v_n) \in V$ if and only if $v_i + v_j = v_k$ for all $i, j, k \in \{1, \ldots, n\}$ with $0 \neq [\mathfrak{n}^E_{r_i}, \mathfrak{n}^E_{r_j}] \subseteq \mathfrak{n}^E_{r_k}$. The subspace V is non-trivial since it contains the solution $(r_1, \ldots, r_n)$. The subset $\mathbb{Z}^n \cap V$ forms a lattice in V since the equations defining V have coefficients in $\mathbb{Q}$.

Let $(z_1, \ldots, z_n)$ be an element of $V \cap \mathbb{Z}^n$. By renaming the subspaces $\mathfrak{n}^E_{r_i} = \tilde{\mathfrak{n}}^E_{z_i}$, we get a positive grading for $\mathfrak{n}^E$ in the sense of Definition 8.1. If the $\mathbb{R}$-grading is positive, so $r_i > 0$ for all $1 \leq i \leq n$, we apply Lemma 4.15 on the open subset of vectors with strictly positive components. It then follows that there exists an element $z = (z_1, \ldots, z_n) \in V \cap \mathbb{Z}^n$ with $z_i > 0$ and thus we find a positive grading in the sense of Definition 8.1. Similarly, starting from a non-trivial and non-negative $\mathbb{R}$-grading, we get a non-trivial and non-negative grading in the sense of Definition 8.1. □

So these definitions of grading on a Lie algebra are completely equivalent for the cases we are interested in. We will use this equivalence between the two definitions later on, as it is often easier to construct an $\mathbb{R}$-grading for a Lie algebra.

Remark 8.6. More general one can define S-gradings for $(S, *)$ any semigroup, i.e. a set equipped with a binary operation $*$ which is associative. An S-grading of a Lie algebra is a decomposition of $\mathfrak{n}^E$ into subspaces $\mathfrak{n}^E_s$ for $s \in S$ such that

$$[\mathfrak{n}^E_s, \mathfrak{n}^E_t] \subseteq \mathfrak{n}^E_{s*t}.$$

For example, a positive grading is nothing else then a grading over the semigroup $\mathbb{N}_0$. In Chapter 9 we will study gradings of the unit group U_K of a number field K. These gradings are closely connected to Anosov Lie algebras.

8.2 Relation between gradings and automorphisms

In this section, we establish a strong relation between gradings on Lie algebras and the existence of certain automorphisms on this Lie algebra. This result is true for any field of characteristic 0, although the proof we give here is given for the rational numbers.

Theorem 8.7. *Let $\mathfrak{n}^E$ a Lie algebra over the field $E \subseteq \mathbb{C}$. The Lie algebra $\mathfrak{n}^E$ has a positive grading if and only if it has an expanding automorphism.*

Note that this result for $E = \mathbb{Q}$ is exactly the equivalence of Theorem 6.13 that wasn't proved in Chapter 6.

For complex Lie algebras, Theorem 8.7 was already described in [36] so this section extends these results to arbitrary subfields of $\mathbb{C}$. This translation between positive gradings and expanding automorphisms allows us to use the theory of linear algebraic groups to study positive gradings.

One implication of Theorem 8.7 is immediate, namely that a positive grading implies the existence of expanding automorphisms. Given a positive grading of

$$\mathfrak{n}^E = \bigoplus_{i>0} \mathfrak{n}_i^E,$$

take any $\mu \in E$ such that $|\mu| > 1$ and consider the automorphism $\varphi_\mu : \mathfrak{n}^E \to \mathfrak{n}^E$ defined as

$$\varphi_\mu(X) = \mu^i X \ \ \forall X \in \mathfrak{n}_i^E.$$

The automorphism φ only has eigenvalues μ^i with $i > 0$ and since $|\mu| > 1$, the automorphism φ_μ is expanding. The hard part is to show that this construction also works the other way around. We will prove the following, slightly stronger theorem.

Theorem 8.8. *Let $\mathfrak{n}^E$ be a Lie algebra over the field $E \subseteq \mathbb{C}$. If $\varphi \in \operatorname{Aut}(\mathfrak{n}^E)$ is an expanding automorphism, then there exists a positive grading on $\mathfrak{n}^E$ such that every automorphism commuting with φ preserves this positive grading.*

In particular, this positive grading is preserved by φ itself. We only give a proof of this theorem in the case $E = \mathbb{Q}$, but this proof generalizes immediately to more general fields. The proof of Theorem 8.8 is based on some facts about algebraic number fields as discussed in Chapter 2.

Take any diagonalizable matrix $A \in \mathrm{GL}(n, \mathbb{Q})$ with characteristic polynomial $p(X) \in \mathbb{Q}[X]$. Denote by $E \subseteq \mathbb{C}$ the minimal splitting field of $p(X)$, which is of course an algebraic number field. Let $v_1, \ldots, v_n$ be a basis of eigenvectors for the matrix A, considered as a linear map $\mathbb{C}^n \to \mathbb{C}^n$, with eigenvalues respectively $\lambda_1, \ldots, \lambda_n$. We define the matrix $N(A) \in \mathrm{GL}(n, \mathbb{C})$ by

$$N(A)(v_i) = N_E(\lambda_i) v_i,$$

so it has the same eigenvectors as A but with eigenvalues $N_E(\lambda_i)$ rather than λ_i. This definition does not depend on the choice of the basis v_i. Let $P \in \mathrm{GL}(n, \mathbb{Q})$ be an invertible matrix over $\mathbb{Q}$ and consider the matrix PAP^{-1}. If $v \in \mathbb{C}^n$ is an eigenvector of A corresponding to the eigenvalue λ, then $P(v)$ is an eigenvector of PAP^{-1} also corresponding to the eigenvalue λ, because

$$\left(PAP^{-1}\right) P(v) = PA(v) = P(\lambda v) = \lambda P(v).$$

This implies that $N(PAP^{-1}) = PN(A)P^{-1}$, showing that $N(A)$ is invariant under change of basis. Note that if B commutes with A, then B also commutes with $N(A)$.

A priori we only know that $N(A) \in \mathrm{GL}(n,\mathbb{C})$, but we show that $N(A) \in \mathrm{GL}(n,\mathbb{Q})$ for every diagonalizable matrix $A \in \mathrm{GL}(n,\mathbb{Q})$. If $p(X) \in \mathbb{Q}[X]$ is a polynomial over $\mathbb{Q}$, we denote by $L(p)$ the companion matrix of the polynomial p. Let L be the primary rational canonical form of A as introduced in Theorem 5.11. This means that

$$L = \begin{pmatrix} L(p_1^{m_1}) & 0 & \dots & 0 \\ 0 & L(p_2^{m_2}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L(p_k^{m_k}) \end{pmatrix}$$

with $m_i > 0$ and the polynomials $p_i(X)$ irreducible over $\mathbb{Q}$. Denote the degree of the polynomials p_i as n_i. Since A is similar over $\mathbb{Q}$ to its primary rational canonical form L, there exists $P \in \mathrm{GL}(n,\mathbb{Q})$ such that $PAP^{-1} = L$.

Now take λ_i a root of the polynomial p_i. Note that $N_E(\lambda_i)$ doesn't depend on the choice of the root λ_i, since all roots of p_i are E-conjugate. It's easy to see then that $N(L)$ is equal to

$$N(L) = \begin{pmatrix} N_E(\lambda_1)\mathbb{1}_{m_1 n_1} & 0 & \dots & 0 \\ 0 & N_E(\lambda_2)\mathbb{1}_{m_2 n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & N_E(\lambda_k)\mathbb{1}_{m_k n_k} \end{pmatrix} \in \mathrm{GL}(n,\mathbb{Q}).$$

From this it follows that $N(A) = P^{-1}N(L)P \in \mathrm{GL}(n,\mathbb{Q})$.

Now assume that $\mathfrak{n}^{\mathbb{Q}}$ is a rational Lie algebra and $\varphi \in \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ an automorphism. Then $N(\varphi)$ is a linear map $\mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ and we claim that it is also a Lie algebra automorphism. Consider the complexification $\mathfrak{n}^{\mathbb{C}}$ of $\mathfrak{n}^{\mathbb{Q}}$, then it suffices to show that $N(\varphi)^{\mathbb{C}} = 1_{\mathbb{C}} \otimes N(\varphi) : \mathfrak{n}^{\mathbb{C}} \to \mathfrak{n}^{\mathbb{C}}$ is a Lie algebra automorphism. By the linearity of the bracket, it is sufficient to show that $N(\varphi)^{\mathbb{C}}$ preserves the brackets between basis vectors for a basis of $\mathfrak{n}^{\mathbb{C}}$. Since there exists a basis of eigenvectors for $\varphi^{\mathbb{C}}$, we show that $N(\varphi)^{\mathbb{C}}$ preserves the bracket of all eigenvectors. Let v_λ and v_μ be two eigenvectors of $\varphi^{\mathbb{C}}$, then $[v_\lambda, v_\mu]$ is an eigenvector of eigenvalue $\lambda\mu$. So

$$\begin{aligned} [N(\varphi)^{\mathbb{C}}(v_\lambda), N(\varphi)^{\mathbb{C}}(v_\mu)] &= N_E(\lambda)N_E(\mu)[v_\lambda, v_\mu] \\ &= N_E(\lambda\mu)[v_\lambda, v_\mu] \\ &= N(\varphi)^{\mathbb{C}}([v_\lambda, v_\mu]), \end{aligned}$$

with E the minimal splitting field for the characteristic polynomial of φ.

To define $N(\varphi)$ we assumed that φ was semisimple. We now extend the definition of N to arbitrary automorphisms of $\mathfrak{n}^{\mathbb{Q}}$. Let $\varphi : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ be an automorphism, then $\varphi = \varphi_s \varphi_u$ for some semisimple automorphism $\varphi_s \in \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ and a unipotent automorphism $\varphi_u \in \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ because of the 'Multiplicative Jordan decomposition' of Theorem 5.15. Every automorphism commuting with φ also commutes with φ_s. We define $N(\varphi)$ as $N(\varphi_s)$ of its semisimple part in the Jordan decomposition.

Alternatively, we could use the generalized eigenspaces of φ to define $N(\varphi)$, see Theorem 5.3. If $\mathfrak{n}^{\mathbb{Q}} = \oplus_{\lambda \in \mathbb{C}} V_\lambda$ is the decomposition into generalized eigenspaces of φ, then $N(\varphi)$ is given by $X \mapsto N_E(\lambda) X$ for every $X \in V_\lambda$.

We conclude this discussion with the following proposition:

Proposition 8.9. *Let $\varphi \in \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ be an automorphism of a rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. Then $N(\varphi) \in \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ is an automorphism with only rational eigenvalues and commuting with every automorphism which commutes with φ. Moreover, if φ is expanding, then $N(\varphi)$ is also expanding.*

Proof. The only thing left to show is the last statement, namely that $N(\varphi)$ is expanding if φ is expanding. The eigenvalues of $N(\varphi)$ are equal to $N_E(\lambda)$ with λ an eigenvalue of φ and E the splitting field of the characteristic polynomial of φ. So it suffices to show that if λ and all its E-conjugates are > 1 in absolute value, then also $N_E(\lambda) > 1$. Since $N_E(\lambda)$ is just the product of the E-conjugates of λ, this follows immediately. $\square$

From this proposition, the proof of Theorem 8.8 with $E = \mathbb{Q}$ now follows directly.

Proof of Theorem 8.8. Let $\varphi : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ be an expanding automorphism. Then $N(\varphi)$ is an expanding automorphism with only eigenvalues in $\mathbb{Q}$. By squaring φ if necessary we can assume that all eigenvalues of $N(\varphi)$ are positive. Since all eigenvalues are rational, the corresponding eigenspaces are rational subspace of $\mathfrak{n}$. Take the $\mathbb{R}$-grading $\mathfrak{n} = \bigoplus_{r \in \mathbb{R}} V_r$ where V_r is the eigenspace of eigenvalue e^r. It's easy to see that this is indeed an $\mathbb{R}$-grading. Since $N(\varphi)$ is expanding, it follows that the grading is positive. The existence of a positive $\mathbb{R}$-grading implies the existence of a positive grading by just renaming the subspaces V_r, as explained above.

If ψ commutes with φ, it also commutes with $N(\varphi)$ and thus it preserves the eigenspaces of $N(\varphi)$. Since the grading is given by eigenspaces of $N(\varphi)$, the last statement follows. $\square$

Remark 8.10. The proof for general fields E is almost identical. If $E \subseteq F$ is a field extension of finite degree n, then there also exist exactly n monomorphisms $E \to \mathbb{C}$ which fix the field E pointwise. For general matrices in $A \in \mathrm{GL}(k, E)$, we can then define the norm $N_E(A)$ which is a matrix with only eigenvalues in E and which commutes with every matrix commuting with A. If $\varphi \in \mathrm{Aut}(\mathfrak{n}^E)$, then $N_E(\varphi)$ will again be an automorphism of $\mathfrak{n}^E$.

The only difference is that E is in general not a subfield of $\mathbb{R}$. This means that we do not find an $\mathbb{R}$-grading directly from the automorphism $N_E(\varphi)$. But the linear map $\tilde{\varphi}$ which maps an eigenvector X corresponding to an eigenvalue λ to

$$\tilde{\varphi}(X) = |\lambda|^2 X$$

has eigenvalues in $E \cap \mathbb{R}$ since $|\lambda|^2 \in E \cap \mathbb{R}$. Moreover, this map is also an automorphism of the Lie algebra.

We now generalize these results to cohopfian $\mathcal{F}$-groups. The translation of expanding maps to the Lie algebra is an expanding automorphism. We start by showing that the translation of a non-trivial self-cover is a partially expanding map.

Assume that N is an $\mathcal{F}$-group which is not cohopfian. This means there exists an injective group morphism $\varphi : N \to N$ which is not surjective. From Section 6.2 we know that φ has characteristic polynomial $p(X) \in \mathbb{Z}[X]$. Since φ is not surjective, Proposition 2.30 implies that $|\det(\varphi)| > 1$. If we decompose $p(X)$ into its $\mathbb{Q}$-irreducible components

$$p(X) = p_1(X) \dots p_l(X),$$

then $p_i(X) \in \mathbb{Z}[X]$ and at least one $p_i(X)$ satisfies $|p_i(0)| > 1$. This means that $N(\varphi)$ is an automorphism with only eigenvalues in $\mathbb{Z}$ and at least one eigenvalue λ with $|\lambda| > 1$. By squaring $N(\varphi)$ if necessary we get that $\mathfrak{n}^{\mathbb{Q}}$ has a partially expanding automorphism.

Theorem 8.11. *Let N be an $\mathcal{F}$-group with corresponding Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. If $\varphi : N \to N$ is an injective group morphism which is not surjective, then $\mathfrak{n}^{\mathbb{Q}}$ has a partially expanding automorphism which commutes with every automorphism commuting with φ.*

So the translation of Theorem 8.8 from expanding maps to non-trivial self-covers is the following.

Theorem 8.12. *Let E be a field of characteristic 0 and $\mathfrak{n}^E$ a Lie algebra over E. The Lie algebra $\mathfrak{n}^E$ has a non-trivial non-negative grading if and only if it has a partially expanding automorphism.*

Proof. Let $\varphi : \mathfrak{n}^E \to \mathfrak{n}^E$ be a partially expanding automorphism and consider the automorphism $N_E(\varphi) \in \mathrm{Aut}(\mathfrak{n}^E)$. We can assume that all eigenvalues of $N_E(\varphi)$ are positive real numbers, just as in the case of expanding automorphisms. The eigenspaces $\mathfrak{n}_r^E$ for the eigenvalues e^r forms an $\mathbb{R}$ grading of $\mathfrak{n}^E$ which is non-negative and non-trival. Every automorphism commuting with φ also commutes with $N_E(\varphi)$ and thus preserves the eigenspaces. □

8.3 Gradings under field extensions

In this section we show that the existence of a positive grading or a non-trivial non-negative grading is invariant under field extensions. Because of Theorem 8.7 and 8.12, this is equivalent to showing that the existence of (partially) expanding automorphisms is invariant under field extensions.

First, we give a proof in the more general case of linear algebraic groups. The tools needed for this result are described in Section 5.3. If $G \leq \mathrm{GL}(n, \mathbb{C})$ is a linear algebraic group, then we call an element $x \in G$ expanding if for all the eigenvalues λ of x, we have $|\lambda| > 1$. We call an automorphism partially expanding if for every eigenvalue λ of x we have $\lambda = 1$ or $|\lambda| > 1$ and x has at least one eigenvalue $\neq 1$.

Theorem 8.13. *Let $K \subseteq L \subseteq \mathbb{C}$ be field extensions and G a linear algebraic K-group. Then $G(K)$ has an expanding element if and only if $G(L)$ has an expanding element.*

Proof. We have a natural inclusion $G(K) \subseteq G(L)$, so if $G(K)$ has an expanding element, also $G(L)$ has an expanding element. For the other implication, it is sufficient to prove it in the case where $L = \mathbb{C}$. Since every power of an expanding element is again expanding, we can also assume that the group G is connected.

Let $x \in G(\mathbb{C})$ be an expanding element. By the multiplicative Jordan decomposition, we can assume that x is semisimple. Every semisimple element of G lies in a maximal torus, so the existence of an expanding element is equivalent to the existence of an expanding element in a maximal torus. Since all maximal tori are conjugate and G contains also a maximal torus defined over K, we can assume that G is a K-torus.

From [10, Section 8.15] it follows that every K-torus G can be written as $G = G_a G_d$ where G_a is an anisotropic subtorus and G_d is a K-split subtorus. Since every K-split torus is conjugated over $\mathrm{GL}(n, K)$ to a K-closed subgroup of $D(n, \mathbb{C})$, we can assume that G_d is a subgroup of $D(n, \mathbb{C})$. Let $x \in G$ be an expanding element and write $x = yz$ with $y \in G_a$ and $z \in G_d$. We show that z is also expanding and hence that the K-split torus G_d contains an expanding element. If not, we can write z up to permutation of the eigenvalues and thus up to conjugation by an element of $\mathrm{GL}(n, \mathbb{Q}) \subseteq \mathrm{GL}(n, K)$ as

$$z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

where Z_1 is a diagonal matrix with only eigenvalues ≤ 1 in absolute value and Z_2 a diagonal matrix with only eigenvalues > 1 in absolute value. Since the torus G_a commutes with z, every element $t \in G_a$ can be written as

$$t = \begin{pmatrix} A_t & 0 \\ 0 & B_t \end{pmatrix}$$

for some invertible matrices A_t and B_t. Consider the character $\chi : G_a \to \mathbb{C}^\times$ defined as $\chi(t) = \det(A_t)$. This character is clearly defined over K and hence must be trivial. In particular, for the element y we have $\det(A_y) = 1$. This would imply that x is not expanding, a contradiction and we conclude that G_d contains an expanding element.

So we are left to prove the theorem in the case where G is a K-split torus, or equivalently, in the case of a K-closed subgroup of $D(n, \mathbb{C})$. Note that the expanding elements form an open subset of $G(\mathbb{R})$ and $G(\mathbb{C}) = G$ for the Euclidean topology. The case $K \subsetneq \mathbb{R}$ is then immediate since $G(K)$ forms a dense subset of $G(\mathbb{C}) = G$ for the Euclidean topology. In the case $K \subseteq \mathbb{R}$, we only have that $G(K)$ is a dense subset of $G(\mathbb{R})$ for the Euclidean topology and therefore it suffices to show that $G(\mathbb{R})$ has an expanding element.

From [10, Section 8.2] it follows that G is defined by character equations, so as the intersection of kernels of characters. Every character $D(n, \mathbb{C}) \to \mathbb{C}^\times$ is of the form

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \mapsto \lambda_1^{k_1} \lambda_2^{k_2} \dots \lambda_n^{k_n}$$

for some $k_i \in \mathbb{Z}$. This implies that if an element

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \in G,$$

then also

$$\begin{pmatrix} |\lambda_1| & 0 & \dots & 0 \\ 0 & |\lambda_2| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |\lambda_n| \end{pmatrix} \in G,$$

since the latter will also satisfy the same character equations. By applying this to an expanding element of G, we get an expanding element of $G(\mathbb{R})$ and this finishes the proof. $\square$

Recall from Chapter 2 that if $\mathfrak{n}^E$ is a Lie algebra, the group $\mathrm{Aut}(\mathfrak{n}^{\mathbb{C}})$ is a linear algebraic E-group with $\mathrm{Aut}(\mathfrak{n}^E)$ as its subgroup of E-rational points. Thus by applying Theorem 8.13 to the automorphism group of a Lie algebra, we have the following consequence:

Theorem 8.14. *Let $E \subseteq F \subseteq \mathbb{C}$ be field extensions and $\mathfrak{n}^E$ a Lie algebra over the field E. Then $\mathfrak{n}^E$ admits an expanding automorphism if and only if $\mathfrak{n}^F$ admits an expanding automorphism. Equivalently, the Lie algebra $\mathfrak{n}^E$ admits a positive grading if and only if the Lie algebra $\mathfrak{n}^F$ admits a positive grading.*

At this point, we want to remark here that Yves Cornulier presented the first proof of this theorem and this over all fields of characteristic 0 in [20, Theorem 1.4]. The approach we present here was developed independently from [20, Theorem 1.4] and uses different methods.

More or less the same proof also works for partially expanding elements instead of expanding elements. In particular we get the following result.

Theorem 8.15. *Let $K \subseteq L \subseteq \mathbb{C}$ be field extensions and G a linear algebraic K-group. Then $G(K)$ has a partially expanding element if and only if $G(L)$ has a partially expanding element.*

Proof. One difference in the case of partially expanding elements is that the set of all partially expanding elements does not form an open subset of $G(\mathbb{R})$ or $G(\mathbb{C})$ for the Euclidean topology. If G is a K-closed subgroup of $D(n, \mathbb{C})$ containing a partially expanding element with the first k eigenvalues equal to 1, then we restrict to the subtorus

$$\tilde{G} = \left\{ \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \in G \;\middle|\; \lambda_1 = \ldots = \lambda_k = 1 \right\} \subseteq G$$

and the proof works for this subtorus $\tilde{G}$.

If we write a partially expanding automorphism $x \in \tilde{G}$ as $x = yz$ with $y \in G_a$ and $z \in G_d$ for an anisotropic torus G_a and a K-split torus G_d, we can show just as in Theorem 8.13 that z has no eigenvalues of absolute value < 1. This implies that the K-split torus G_d has a partially expanding element. □

By applying the result to Lie algebras, we get the following.

Theorem 8.16. *Let $\mathfrak{n}^E$ be a Lie algebra over a field $E \subseteq \mathbb{C}$ and $E \subseteq F \subseteq \mathbb{C}$ a field extension. Then $\mathfrak{n}^E$ admits a partially expanding automorphism if and only if $\mathfrak{n}^F$ admits a partially expanding automorphism. Equivalently, $\mathfrak{n}^E$ admits a non-trivial and non-negative grading if and only if $\mathfrak{n}^F$ admits a non-trivial and non-negative grading.*

8.4 Gradings preserved by automorphisms

In this section, we study gradings preserved by certain automorphisms of the Lie algebra. The strongest result we will present states that given a positive grading and a reductive subgroup of automorphisms, we find always find a positive grading which is preserved by all these automorphisms. Note that every finite group and every diagonalizable groups of automorphisms satisfies this property, since they only contain semisimple elements. Again, we first consider the more general case of linear algebraic groups:

Theorem 8.17. *Let G be a linear algebraic K-group with an expanding element and $H \subseteq G(K)$ a subgroup contained in a reductive subgroup of G. Then there exists an expanding element of $G(K)$ which commutes with every element of H.*

Proof. The elements of G which commute with every element of H form a K-closed subgroup of G, so because of Theorem 8.13 it suffices to show that G contains an expanding element which commutes with every element of H.

Since H is a subgroup of a reductive subgroup of G, there exists a Levi subgroup L of G such that $H \subseteq L$. Every expanding semisimple element of G also lies in a Levi subgroup. Because all Levi subgroups are conjugated, we know that if the group G has an expanding element, the Levi subgroup L (and therefore also L^0) has an expanding element.

The group L^0 is reductive and thus L^0 can be written as

$$L^0 = Z \cdot \mathcal{D}L^0$$

with Z the connected center of L^0. Take x an expanding element of L^0, then we can write it as $x = zy$ with $y \in \mathcal{D}L^0$ and $z \in Z$. Just as above in Theorem 8.13, we claim that z is an expanding element. If z is not expanding, we can write z up to conjugation in $\mathrm{GL}(n, \mathbb{C})$ as

$$z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

where Z_1 is an invertible matrix with only eigenvalues ≤ 1 in absolute value and Z_2 an invertible matrix with only eigenvalues > 1 in absolute value. Since z lies in the center of L^0, every element $t \in L^0$ is of the form

$$t = \begin{pmatrix} A_t & 0 \\ 0 & B_t \end{pmatrix}$$

for some invertible matrices A_t and B_t. Consider now the character $\chi : L^0 \to \mathbb{C}^\times$ given by $\chi(t) = \det(A_t)$. Since $y \in \mathcal{D}L^0$, we have $\chi(y) = 1$ and this is a contradiction since $x = zy$ is expanding. We deduce that the connected center of L^0 contains an expanding element.

Since L^0 has finite index in L, there exist elements $l_1, \ldots, l_k \in L$ such that $L = l_1 L^0 \sqcup \ldots \sqcup l_k L^0$. The subgroup Z is a characteristic subgroup in L, so we have $l_i Z l_i^{-1} = Z$ for all $1 \leq i \leq k$. Fix an expanding element $z \in Z$, then the element

$$z_0 = \prod_{1 \leq i \leq k} l_i z l_i^{-1} \in Z$$

commutes with every element l_i with $1 \leq i \leq k$ and therefore also with every element of L. The group H is a subgroup of L and thus z_0 commutes with every element of H. Since all the elements $l_i z l_i^{-1} \in Z$ are expanding and commute, it follows that z_0 is expanding. This finishes the proof. $\square$

By applying this theorem to the linear algebraic group $\mathrm{Aut}(\mathfrak{n}^{\mathbb{C}})$ of a rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ and taking H a finite subgroup, we get:

Theorem 8.18. *Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational Lie algebra which admits an expanding automorphism and $H \subseteq \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ a finite subgroup. Then there exists an expanding automorphism of $\mathfrak{n}^{\mathbb{Q}}$ which commutes with every element of H. Equivalently, if $\mathfrak{n}^{\mathbb{Q}}$ admits a positive grading, it also admits a positive grading which is preserved by H.*

The same result holds for partially expanding maps in linear algebraic groups with essentially the same proof.

Theorem 8.19. *Let G be a linear algebraic K-group with a partially expanding element and $H \subseteq G(K)$ a subgroup contained in a reductive subgroup of G. Then there exists an expanding element of $G(K)$ which commutes with every element of H.*

Applied to Lie algebras, we get the following result.

Theorem 8.20. *Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational Lie algebra which admits a partially expanding automorphism and $H \subseteq \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$ a finite subgroup. Then there exists a partially expanding automorphism of $\mathfrak{n}^{\mathbb{Q}}$ which commutes with every element of H. Equivalently, if $\mathfrak{n}^{\mathbb{Q}}$ admits a non-negative and non-trivial grading, it also admits a non-negative and non-trivial grading which is preserved by H.*

These results were proved independently and by different methods in [20, Corollary 3.26].

8.5 Applications

In this section, we give some applications of the main results of this chapter. One application is the main result of Part II, as described in the introduction. We also show how we can use Theorem 8.18 to study Anosov diffeomorphisms.

Algebraic characterization of expanding maps

In Chapter 6, more specifically in Theorems 6.15 and 6.18, we gave the relation between expanding maps on infra-nilmanifolds and positive gradings on rational Lie algebras. By combining the results about gradings on Lie algebras with this relation, we get the algebraic description of infra-nilmanifolds admitting an expanding map as discussed in the introduction of Part II.

Theorem 8.21. *Let $\Gamma\backslash G$ be an infra-nilmanifold modeled on the Lie group G with corresponding Lie algebra $\mathfrak{g}$. Then the following are equivalent:*

(1) *The infra-nilmanifold $\Gamma\backslash G$ admits an expanding map;*

(2) *The Lie algebra $\mathfrak{g}$ has a positive grading;*

(3) *The Lie group G has an expanding automorphism.*

Proof. The proof follows from combining Theorem 6.15, Theorem 8.14 and Theorem 8.18. □

Similarly, we have the following result for the existence of non-trivial self-covers on infra-nilmanifolds.

Theorem 8.22. *Let $\Gamma \backslash G$ be an infra-nilmanifold modeled on a Lie group G with corresponding Lie algebra $\mathfrak{g}$. Then the following are equivalent:*

(1) *The infra-nilmanifold $\Gamma \backslash G$ admits a non-trivial self-cover;*

(2) *The group Γ is not cohopfian;*

(3) *The Lie algebra $\mathfrak{g}$ has a non-trivial non-negative grading;*

(4) *The Lie group G has a partially expanding automorphism.*

Proof. The proof is immediate by combining Theorem 6.18, Theorem 8.16 and Theorem 8.20. $\square$

Theorem 8.21 and Theorem 8.22 form Main Theorem 1 and 2 of this second part of the thesis.

In Section 9.3.5 we will use Theorem 8.18 and Theorem 8.21 to give the first example of a nilmanifold admitting an Anosov diffeomorphism but no non-trivial self-cover and so also no expanding map. Since constructing a nilmanifold admitting an Anosov diffeomorphism is in general a hard task, we first need the machinery of Chapter 9 before giving this example.

Commuting expanding maps and Anosov diffeomorphisms

As another application of Theorem 8.17 we can show that there always exists commuting expanding maps and Anosov diffeomorphisms, if both of them exist on the infra-nilmanifold.

Theorem 8.23. *Let $\Gamma \backslash G$ be an infra-nilmanifold admitting an expanding map and an Anosov diffeomorphism. Then there exists an expanding map and an Anosov diffeomorphism on $\Gamma \backslash G$ which commute.*

Proof. Let N be the Fitting subgroup of Γ and $N^{\mathbb{Q}}$ the corresponding radicable Hull. Denote the rational holonomy representation by $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$ where F is the holonomy group of Γ. If $\Gamma \backslash G$ admits an Anosov diffeomorphism, then Theorem 3.36 implies that there exists a hyperbolic automorphism φ on $\mathfrak{n}^{\mathbb{Q}}$, such that φ has characteristic polynomial in $\mathbb{Z}[X]$, $|\det(\varphi)| = 1$ and φ commutes with every element of $H \subseteq \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$. Because of the multiplicative jordan decomposition, we can assume φ to be semisimple.

Take T the smallest linear algebraic subgroup of $\mathrm{Aut}(\mathfrak{n}^{\mathbb{C}})$ which contains φ. The group T will also commute with every element of H and the group HT forms a reductive subgroup of $\mathrm{Aut}(\mathfrak{n}^{\mathbb{C}})$. By Theorem 8.17, we conclude that there exists a positive grading $\mathfrak{n}^{\mathbb{Q}} = \bigoplus_{i>0} \mathfrak{n}_i^{\mathbb{Q}}$ on $\mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ which is preserved by φ and every element of H.

From the proof of Theorem 3.36, it follows that some power φ^k of φ satisfies $\varphi^k \Gamma \varphi^{-k} = \Gamma$ and thus induces a hyperbolic affine infra-nilmanifold automorphism on $\Gamma \backslash G$. Let p be any prime and consider the expanding automorphisms $\psi_p : \mathfrak{n} \to \mathfrak{n}$ which are given by $\psi_p(x) = p^i x$ for all $x \in \mathfrak{n}_i^{\mathbb{Q}}$. By the proof of Theorem 6.15 we know that there exists a prime p and $l > 0$ such that $\psi_p^l \Gamma \psi_p^{-l} \subseteq \Gamma$. The expanding affine infra-nilmanifold endomorphism induced by ψ_p^l commutes with the Anosov diffeomorphisms induced by φ^k and this ends the proof. $\square$

One application of Theorem 8.23 is a simplification of Theorem 3.31. Indeed, the conditions of this result can be relaxed to the existence of an expanding map on the infra-nilmanifold or thus of a positive grading on the Lie algebra of the covering Lie group.

Theorem 8.23 is also true for non-trivial self-covers on infra-nilmanifolds by replacing positive grading by non-trivial and non-negative grading in the proof.

Theorem 8.24. *Let $\Gamma \backslash G$ be an infra-nilmanifold admitting a non-trivial self-cover and an Anosov diffeomorphism. Then there exists a non-trivial self-cover and an Anosov diffeomorphism on $\Gamma \backslash G$ which commute.*

Applications to Lie algebras

We already used the results of this chapter in Section 6.5 to show that some of the examples there are of minimal nilpotency class or of minimal dimension. Since it is easier to classify the complex Lie algebras then the rational Lie algebras, it is easier to check the existence of positive grading on complex Lie algebras.

As another application, see Remark 6.28, it is easier to check that certain Lie algebras do not admit an expanding automorphism.

Example 8.25. Let $\mathfrak{n}^{\mathbb{Q}}$ be the Lie algebra of Example 6.27, so $\mathfrak{n}^{\mathbb{Q}}$ has basis $X_1, \ldots, X_7$ and Lie bracket given by the relations

$$\begin{aligned}
&[X_1, X_2] = X_3 \qquad &&[X_2, X_3] = X_5 \qquad &&[X_2, X_6] = X_7\\
&[X_1, X_3] = X_4 &&[X_2, X_4] = X_6 &&[X_3, X_5] = -X_7\\
&[X_1, X_5] = X_6 &&[X_2, X_5] = X_7.
\end{aligned}$$

We left it as an exercise to check that $\mathfrak{n}^{\mathbb{Q}}$ does not have expanding automorphisms. By using Theorem 8.18 this now follows directly.

Let $\psi \in \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ be the automorphism of order 2 such that

$$\psi(X_i) = X_i \;\; i \in \{1,5,6,7\}$$

$$\psi(X_i) = -X_i \;\; i \in \{2,3,4\}.$$

From the relations above it follows that ψ is indeed an automorphism of $\mathfrak{n}^{\mathbb{Q}}$. By Theorem 8.18 it follows that if $\mathfrak{n}^{\mathbb{Q}}$ has an expanding automorphism, then it also has an expanding automorphism commuting with ψ.

Assume that φ is such an expanding automorphism commuting with ψ. Let $\bar{\varphi} : {}^{\mathfrak{n}^{\mathbb{Q}}}\!/_{\gamma_2(\mathfrak{n}^{\mathbb{Q}})} \to {}^{\mathfrak{n}^{\mathbb{Q}}}\!/_{\gamma_2(\mathfrak{n}^{\mathbb{Q}})}$ be the automorphism induced by φ. Since φ commutes with ψ, $\bar{\varphi}$ is given by a diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

for the natural basis given by $X_1 + \gamma_2(\mathfrak{n}^{\mathbb{Q}}), X_2 + \gamma_2(\mathfrak{n}^{\mathbb{Q}})$. From the relations, it follows that X_7 is an eigenvector of both eigenvalue $\lambda_1\lambda_2^3$ and $\lambda_1^2\lambda_2^3$. This implies that $\lambda_1 = 1$ and thus φ is not expanding.

The other examples of Section 6.5 can be treated in the same way. So if we know some automorphisms of finite order on the Lie algebra, Theorem 8.18 allows us to check more easily if the Lie algebra has a positive grading or not.

Part III

Anosov diffeomorphisms

In Part II of this thesis we showed that the existence of an expanding map on an infra-nilmanifold $\Gamma\backslash G$ depends only on the covering Lie group G. The first step of this proof was the relation between the rational holonomy representation of Γ and expanding maps as discussed in Chapter 6. For Anosov diffeomorphism exactly the same arguments give us the following criterion depending on the rational holonomy representation, as we already mentioned in Chapter 3.

Theorem 3.36. *Let $\Gamma\backslash G$ be an infra-nilmanifold with associated rational holonomy representation $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$. Then $\Gamma\backslash G$ admits an Anosov diffeomorphism if and only if there exists a hyperbolic integer-like automorphism $\varphi \in \mathrm{Aut}(N^{\mathbb{Q}})$ which commutes with every element of $\rho(F)$.*

In fact, every diffeomorphism induces an automorphism on the fundamental group which has determinant equal to ± 1. Hence it suffices to apply the more restrictive Theorem 6.5 in the proof, avoiding the prime numbers of Theorem 6.7 which was the one of the main results in Chapter 6.

Contrary to the situation of expanding maps, the existence of an Anosov diffeomorphism does not only depend on the covering Lie group. There are many examples of nilmanifolds $\Gamma_1\backslash G$ and $\Gamma_2\backslash G$ modeled on the same nilpotent Lie group G such that $\Gamma_1\backslash G$ admits an Anosov diffeomorphism and $\Gamma_2\backslash G$ doesn't. We give two examples illustrating the problems that can occur.

Example 1. Consider the Lie group $G = H_3(\mathbb{R}) \oplus H_3(\mathbb{R})$ with lattice $N = H_3(\mathbb{Z}) \oplus H_3(\mathbb{Z})$. A computation shows that $N\backslash G$ doesn't admit an Anosov diffeomorphism since every integer-like automorphism of $N^{\mathbb{Q}}$ has an eigenvalue of absolute value 1. On the other hand, there are lattices of G such that the corresponding nilmanifold does admit an Anosov diffeomorphism.

Example 2. Consider the torus $\mathbb{T}^2$ and the Klein bottle $\mathbb{K}^2$, which are both infra-nilmanifolds modeled on the abelian Lie group $\mathbb{R}^2$. The rational holonomy representation of the Klein bottle is equal to

$$\mathbb{Z}_2 \to \mathrm{GL}(2,\mathbb{Q}) : \pm 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

The only integer-like automorphisms commuting with this representation are diagonal matrices with eigenvalues ± 1. This implies that $\mathbb{K}^2$ doesn't admit an Anosov diffeomorphism, whereas $\mathbb{T}^2$ does support an Anosov diffeomorphism, e.g. Arnold's cat map.

The examples show that both intermediate results of Chapter 8 are false for Anosov diffeomorphism. Recall that the first step of that chapter is showing that the existence of an expanding automorphism is invariant under field extensions.

The second step is then to find an expanding automorphism commuting with a given finite subgroup of automorphisms starting from an expanding automorphism. Since both results are false for Anosov diffeomorphisms, it is more complicated to study their existence on infra-nilmanifolds.

The idea of this third part of the thesis is to start from a fixed Lie group G and study Anosov diffeomorphisms on infra-nilmanifolds modeled on G. If $\mathfrak{g}$ is the corresponding Lie algebra of G, Theorem 3.36 gives us a tool to determine which infra-nilmanifolds support an Anosov diffeomorphism by looking at rational subalgebras of $\mathfrak{g}$ and finite groups of automorphisms of this subalgebra. Hence the first question is always finding rational forms of $\mathfrak{g}$ which admit a hyperbolic integer-like automorphism. The Lie algebras satisfying this property are called Anosov Lie algebras and they are studied in Chapter 9.

More specific, Chapter 9 gives a new method for constructing Anosov Lie algebras, generalizing the explicit examples which were already known in literature. Although the exact statement of this result, see Theorem 9.14, is rather technical, it answers many open existence questions about Anosov Lie algebras. One instance is the construction of a nilmanifold admitting an Anosov diffeomorphism but no expanding map, which is the first example of this type. This new construction implies the existence of Anosov Lie algebras of minimal type or having an Anosov automorphism of minimal signature. We also correct some errors in the classification of Anosov Lie algebras up to dimension 8 which was given in [75].

As there are only very few classes of nilmanifolds for which the existence of an Anosov diffeomorphism is completely described, the situation for infra-nilmanifolds is even more complicated. Examples of such classes with a complete description are nilmanifolds modeled on free nilpotent Lie groups, on Lie groups of dimension ≤ 8 and on Lie groups associated to graphs. Starting from these classifications, Chapter 10 determines which infra-nilmanifolds modeled on free nilpotent Lie groups admit an Anosov diffeomorphism, depending only on how the abelianized rational holonomy representation splits over the rationals $\mathbb{Q}$ and the reals $\mathbb{R}$.

Chapter 9

Anosov Lie algebras

As explained in the introduction, Anosov Lie algebras are the rational nilpotent Lie algebras corresponding to nilmanifolds admitting an Anosov diffeomorphism. This chapter describes a new method for constructing Anosov Lie algebras, which generalizes and simplifies the known methods in the literature. A lot of the examples of Anosov Lie algebras are explicitly constructed and the main result of this chapter shows how to avoid all computations needed for these examples. It also allows us to construct more complicated instances of Anosov automorphisms, for example Anosov automorphisms of minimal signature or living on a Lie algebra of minimal type.

The first section starts by giving the necessary background to study these Lie algebras, together with some recent results and open existence questions. Next, we give our main result in Theorem 9.14. Although the statement of this method is rather technical, it has many interesting applications which we give in the last section. Among these applications, there is a correction to the classification of Anosov Lie algebras up to dimension 8 and the construction of a nilmanifold admitting an Anosov diffeomorphism but no expanding map.

9.1 Background and open questions

In this first section, we discuss Anosov Lie algebras, which are nilpotent Lie algebras admitting an Anosov automorphism. We also introduce the signature of an Anosov automorphism and the type of a nilpotent Lie algebra, including some existence questions about Anosov automorphisms we will answer in this chapter.

Theorem 3.36 implies that a nilmanifold $\Gamma\backslash G$ admits an Anosov diffeomorphism if and only if there exists a hyperbolic and integer-like automorphism of $\mathfrak{n}^{\mathbb{Q}}$. Recall that a matrix is called integer-like if its characteristic polynomial has coefficients in $\mathbb{Z}$ and its determinant has absolute value 1. Since the eigenvalues and characteristic polynomial of a matrix are invariant under conjugation, the properties hyperbolic and integer-like are invariant under change of basis. An automorphism is hyperbolic respectively integer-like if the matrix representation of the automorphism for some (and thus for every) basis has the same property. This motivates the following definition:

Definition 9.1. An automorphism $\alpha \in \operatorname{Aut}(\mathfrak{n}^E)$ of a Lie algebra $\mathfrak{n}^E$ over some field $E \subseteq \mathbb{C}$ is called **Anosov** if it is hyperbolic and integer-like. A rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ with an Anosov automorphism is called an **Anosov Lie algebra**.

Thus a classification of all nilmanifolds admitting an Anosov diffeomorphism is equivalent to a classification of all Anosov Lie algebras. Theorem 8.3 shows that every Anosov Lie algebra is necessarily nilpotent, by considering the grading induced from the eigenspaces of a semisimple Anosov automorphism.

Theorem 9.14 will give us a very general way of constructing Anosov Lie algebras. By using the low-dimensional classification of Anosov Lie algebras, we can describe the isomorphism class for some of those Lie algebras in Section 9.3.2, but it will be hard to do the same in general.

On the other hand, there will be some properties of the Lie algebra or the Anosov automorphisms which follow immediately from the construction, e.g. the signature of the Anosov automorphism and the type of the Lie algebra, properties which we introduce below.

The definition of the signature of an Anosov diffeomorphism $f : M \to M$ makes use of the df-invariant splitting $TM = E^s \oplus E^u$, see Section 3.3.

Definition 9.2. The **signature** $\operatorname{sgn}(f)$ of an Anosov diffeomorphism $f : M \to M$ is defined as the set $\{\dim_{\mathbb{R}} E^s, \dim_{\mathbb{R}} E^u\}$.

We define the signature as a set rather than as an ordered pair since the order of the elements does not play a role. Indeed, the inverse of an Anosov diffeomorphism is again Anosov with the bundles E^s and E^u interchanged. For Anosov automorphisms $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ on a rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$, the signature is defined as the set $\{p, q\}$ where p is the number of eigenvalues with absolute value < 1 and q the number of eigenvalues of absolute value > 1. Note that

Theorem 6.1 and Theorem 6.5 imply that there exists an Anosov diffeomorphism f with signature $\operatorname{sgn}(f)$ if and only if there exists an Anosov automorphisms with signature $\operatorname{sgn}(f)$ on the corresponding Lie algebra.

The following question is stated in [75].

Question 9.1. Does there exist an Anosov automorphism on a non-abelian Lie algebra with signature $\{2, k\}$ for some $k \in \mathbb{N}$?

In Section 9.3.3 it is shown that $\min(\operatorname{sgn}(f)) \geq c$ where c is the nilpotency class of the Lie algebra. So a more general question is the existence of Anosov automorphisms which attain this lower bound for the signature. The main theorem gives us a positive answer in Section 9.3.3.

As a consequence of a low-dimensional classification of Anosov automorphisms, the following is stated as a corollary in [75]:

Claim 9.1. Let $\Gamma\backslash G$ be a nilmanifold of dimension ≤ 8 which admits an Anosov diffeomorphism. Then $\Gamma\backslash G$ is a torus or the dimension is 6 or 8 and the signature is $\{3, 3\}$ or $\{4, 4\}$ respectively.

Examples illustrating that some cases were overlooked in this corollary are given in Section 9.3.2 with signature $\{2, 4\}$ and $\{3, 5\}$. A complete list of Anosov Lie algebras admitting such signatures is given and thus this section forms a correction to the result of [75].

Let $\mathfrak{n}^E$ is a Lie algebra over a field E, and consider its lower central series $\gamma_i(\mathfrak{n}^E)$. The type of a Lie algebra gives us information about the quotients of the lower central series.

Definition 9.3. The **type** of a nilpotent Lie algebra $\mathfrak{n}^E$ of nilpotency class c is defined as the c-tuple $(n_1, \ldots, n_c)$, where $n_i = \dim_E \gamma_i(\mathfrak{n}^E) / \gamma_{i+1}(\mathfrak{n}^E)$.

If $\mathfrak{n}^{\mathbb{Q}}$ is an Anosov Lie algebra of type $(3, n_2, \ldots, n_c)$, then we will show that $3 \mid n_i$ for every $i \in \{2, \ldots, c\}$. Combining this result with [74, Proposition 2.3.] we get that every Anosov Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ of type $(n_1, \ldots, n_c)$ satisfies one of the following:

(1) $\mathfrak{n}^{\mathbb{Q}}$ is abelian or

(2) $n_1 \geq 4$ and $n_i \geq 2$ for all $i \in \{2, \ldots, c\}$ or

(3) $n_1 = n_2 = 3$ and $3 \mid n_i$ for all $i \in \{3, \ldots, c\}$.

In [72] it was proved that the lower bound of (2) occurs for every c and the main theorem shows that also the lower bound of (3) is attained for every c, answering the following question of [74].

Question 9.2. Does there exist a c-step Lie algebra of type $(3, \ldots, 3)$ of nilpotency class $c \geq 3$ which is Anosov?

An Anosov Lie algebra which attains one of the lower bounds (ii) or (iii) for the type will be called of minimal type. In fact, in Section 9.3.3 we will show that Question 9.2 has a positive answer even if we replace 3 by any integer $n > 2$.

To end this first section, we give a short summary of the known results about Anosov Lie algebras. In a few cases, a complete classification is given, for example in low-dimensional cases (see [74]) and in the case of free nilpotent Lie algebras (see [21, 31]). All other examples of Anosov Lie algebras are explicitly constructed, following two completely different methods.

The first method is top-down and starts from a free nilpotent Lie algebra. One then takes a quotient Lie algebra by dividing out an ideal of the free nilpotent Lie algebra. Every automorphism of the free nilpotent Lie algebra which maps the ideal to itself then induces an automorphism on the quotient Lie algebra. This method is based on the work of L. Auslander and J. Scheuneman in [5]. In [22, 33, 51, 89] some new examples of Anosov diffeomorphisms on nilmanifolds were constructed in this way.

The second methods is bottom-up and starts from a Lie algebra $\mathfrak{n}^E$ over some number field E given by generators and relations. A rational form of this Lie algebra is then constructed together with a hyperbolic automorphism mapping this rational form to itself. The hard step is to construct an explicit basis of $\mathfrak{n}^E$ such that the structure constants are rational and the matrix representation of the automorphism has rational entries. Note that in the papers [72, 74, 75, 81], it is checked for each example separately that the given set of vectors is in fact a basis, that the structure constants do lie in $\mathbb{Q}$ and that the matrix of f with respect to this basis has entries in $\mathbb{Q}$. The computations needed for these steps are rather cumbersome and somewhat time-consuming.

The main result of this chapter is to generalize this latter construction in Theorem 9.14. This theorem avoids all the computations and allows us to construct more complicated examples of Anosov automorphisms. It seems plausible that all questions about the existence of Anosov automorphisms of specific signature on a Lie algebra of specific type can be answered in this way. The disadvantage of this method is that in general it is hard to describe the rational Lie algebra in terms of a basis and relations on this basis.

9.2 A new method for constructing Anosov Lie algebras

The examples of Anosov Lie algebras in the papers [72, 75, 81] are constructed from a nilpotent Lie algebra $\mathfrak{n}^E$ over some field E given by its decomposition into eigenspaces of an Anosov automorphism. An explicit basis is then constructed which is 'symmetric' under the action of the Galois group of E. Instead of working with a basis, we rather use this 'symmetric' property of the basis as the definition of a rational form of the Lie algebra. We will make this statement precise in Section 9.2.1. For automorphisms, a similar 'symmetric' property can be defined such that the automorphism induces an automorphism on the rational form. In this way, no explicit calculations are needed and Theorem 9.14 allows us to easily show the existence of several new Anosov Lie algebras. This method further generalizes the work of [89] which starts from polynomials instead of field extensions.

The proof of this main result falls apart in three different steps. The first step is to use this 'symmetric' property to construct a rational vector space of $\mathfrak{n}^E$ and study when this subspace is a rational form of $\mathfrak{n}^E$ as vector space over E. Next, we show that this rational subspace is always a subalgebra of the Lie algebra $\mathfrak{n}^E$. Finally the same techniques are used to construct automorphisms on this rational forms.

First let us fix some notations for this section. Let E be a subfield of $\mathbb{C}$ such that E has finite degree over $\mathbb{Q}$. We will always assume that our field extensions are Galois. Just as for Lie algebras, a vector space V over E will be denoted by V^E. All the vector spaces and Lie algebras we consider are finite dimensional. If $F \supseteq E$ is a field extension, then we can consider the vector space $F \otimes_E V^E$ which we will denote as V^F. A rational subspace $W^{\mathbb{Q}} \subseteq V^E$ is called a rational form if some (and hence every) basis of $W^{\mathbb{Q}}$ over $\mathbb{Q}$ is also a basis of V^E over E. If $\mathfrak{n}^E$ is a Lie algebra over E, we call a rational subalgebra $\mathfrak{m}^{\mathbb{Q}} \subseteq \mathfrak{n}^E$ a rational form if it is a rational form seen as subspace of $\mathfrak{n}^E$ as vector space.

9.2.1 Constructing a rational form of a vector space

Instead of focusing on the basis of the rational form of a vector space, we focus on the defining property of the rational form as being 'symmetric' under the action of the Galois group. For this we first introduce the action of the Galois group on a vector space and define for each representation a rational subspace.

Consider the natural right action of $\mathrm{Gal}(E, \mathbb{Q})$ on the field E, given by

$$\forall \sigma \in \mathrm{Gal}(E, \mathbb{Q}), \forall x \in E : x^\sigma = \sigma^{-1}(x).$$

By defining the action component-wise, there is also a natural right action on the vector space E^m. Note that the relations

$$(\sigma(\lambda)x)^\sigma = \lambda x^\sigma \text{ and } (x+y)^\sigma = x^\sigma + y^\sigma \tag{9.1}$$

hold for all $x, y \in E^m$, $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ and $\lambda \in E$. Let $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{GL}(m, E)$ be a representation, then it follows immediately from equation (9.1) that the subset defined as

$$V_\rho^{\mathbb{Q}} = \{v \in E^m \mid \forall \sigma \in \mathrm{Gal}(E, \mathbb{Q}), \rho_\sigma(v) = v^\sigma\} \tag{9.2}$$

is a rational subspace of E^m. This subspace is already close to being a rational form of E^m in the sense of the following lemma:

Lemma 9.4. *If a set of vectors $v_1, \ldots, v_k$ of $V_\rho^{\mathbb{Q}}$ is linearly independent over $\mathbb{Q}$, then this set is also linearly independent over E as vectors of E^m.*

Proof. Assume that the lemma does not hold and take vectors $v_1, \ldots, v_k$ of $V_\rho^{\mathbb{Q}}$ with k minimal which are linearly independent over $\mathbb{Q}$ but contradict the statement. This means that there exists $x_i \in E$ such that

$$\sum_{i=1}^{k} x_i v_i = 0. \tag{9.3}$$

From the minimality of k it follows that $x_1 \neq 0$ and thus by multiplying this equation by x_1^{-1} we can assume that $x_1 = 1$. Since k is minimal, it follows that the x_i are the unique elements of E such that $x_1 = 1$ and equation (9.3) is true.

If we apply the map ρ_σ to the equation, we get that

$$0 = \rho_\sigma \left(\sum_{i=1}^{k} x_i v_i \right) = \sum_{i=1}^{k} x_i \rho_\sigma(v_i) = \sum_{i=1}^{k} x_i v_i^\sigma = \left(\sum_{i=1}^{k} \sigma(x_i) v_i \right)^\sigma$$

because of (9.1). We also have that

$$\sum_{i=1}^{k} \sigma(x_i) v_i = 0.$$

Minimality of k and the fact that $\sigma(x_1) = \sigma(1) = 1$ imply that $\sigma(x_i) = x_i$ for all i. Because this statement holds for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$, we conclude that the coefficients x_i lie in $\mathbb{Q}$ and thus we get a contradiction since v_i was a set of linearly independent vectors over $\mathbb{Q}$. $\square$

The lemma shows that the rational subspace $V_\rho^{\mathbb{Q}}$ is a rational form of E^m if its dimension is maximal. This motivates the following definition:

Definition 9.5. A representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{GL}(m, E)$ is called **Galois compatible** (abbreviated as GC) if and only if $\dim_{\mathbb{Q}}(V_\rho^{\mathbb{Q}}) = m$. Equivalently, the representation ρ is GC if and only if $E \otimes V_\rho^{\mathbb{Q}} = E^m$ or if $V_\rho^{\mathbb{Q}}$ is a rational form of E^m.

The trivial representation is the easiest example of a GC representation, since in that case $V_\rho^{\mathbb{Q}} = \mathbb{Q}^m$. The next examples we consider are the regular representation and more generally, representations given by permutations matrices.

Example 9.6. Let ρ be the regular representation of $\mathrm{Gal}(E, \mathbb{Q})$, i.e. take the vector space over E spanned by the basis $\{v_\sigma \mid \sigma \in \mathrm{Gal}(E, \mathbb{Q})\}$ and the representation ρ induced by the relations $\rho_\tau(v_\sigma) = v_{\tau\sigma}$ for all $\tau, \sigma \in \mathrm{Gal}(E, \mathbb{Q})$. Every element v of the rational vector space $V_\rho^{\mathbb{Q}}$ is given by

$$v = \sum_{\sigma \in \mathrm{Gal}(E,\mathbb{Q})} \sigma(x) v_\sigma$$

for some $x \in E$. This is a rational vector space of dimension $[E : \mathbb{Q}]$, which is also the order of the group $\mathrm{Gal}(E, \mathbb{Q})$ and thus the dimension of the regular representation ρ. This shows that ρ is indeed GC.

Example 9.7. First we recall what a permutation matrix is. For every permutation $\pi \in S_n$, there exists a matrix $K_\pi \in \mathrm{GL}(n, \mathbb{Q})$ defined as

$$(K_\pi)_{ij} = \begin{cases} 1 & j = \pi(i) \\ 0 & \text{otherwise} \end{cases}$$

Every matrix of the form K_π for some $\pi \in S_n$ is called a permutation matrix.

Let $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{GL}(n, \mathbb{Q})$ be given by permutation matrices, meaning that ρ_σ is a permutation matrix for every $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$. Write the standard basis of E^n as $e_1, \ldots, e_n$ and note that ρ also acts on the basis $\alpha = \{e_i \mid 1 \leq i \leq n\}$. By decomposing the basis α into its orbits, we can assume without loss of generality that the action is transitive, i.e. for every $i \in \{1, \ldots, n\}$, there exists a $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ such that $\rho_\sigma(e_1) = e_i$. In the Appendix we study the relation between actions of groups on finite sets and subgroups of finite index.

For every $1 \leq i \leq n$, fix an element $\sigma_i \in \mathrm{Gal}(E, \mathbb{Q})$ with $\sigma_i(e_1) = e_i$. Note that every element $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ such that $\sigma(e_1) = e_i$ is of the form $\sigma = \sigma_i h$ for some $h \in H$. Let H be the subgroup of $\mathrm{Gal}(E, \mathbb{Q})$ which fixes the vector e_1 and E' the subfield of E which is fixed by the group H. The group H is a subgroup of index n in the group $\mathrm{Gal}(E, \mathbb{Q})$ as follows from the Appendix and thus the field E' has degree n over $\mathbb{Q}$. Let $x \in E'$, then every element

$$v = \sum_{i=1}^{n} \sigma_i(x) e_i$$

is an element of $V_\rho^{\mathbb{Q}}$. Since E' has degree n over $\mathbb{Q}$, this is a vector space of dimension n over $\mathbb{Q}$ and thus ρ is GC.

Of course, there are also examples of representations which are not GC.

Example 9.8. Consider the field $E = \mathbb{Q}(i)$ with complex conjugation $\sigma : E \to E$ as a field automorphism, meaning that σ is defined by $\sigma(i) = -i$. The Galois group of E over $\mathbb{Q}$ is equal to

$$\mathrm{Gal}(E, \mathbb{Q}) = \{\mathbb{1}_E, \sigma\} \simeq \mathbb{Z}_2.$$

Now consider the representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{GL}(2, E)$ defined as

$$\rho(\sigma) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The subspace $V_\rho^{\mathbb{Q}}$ is given by the condition

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} \in V_\rho^{\mathbb{Q}} &\iff \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sigma^{-1}(x) \\ \sigma^{-1}(y) \end{pmatrix} = \begin{pmatrix} \sigma(x) \\ \sigma(y) \end{pmatrix} \\
&\iff \begin{cases} -iy = \sigma(x) \\ ix = \sigma(y) \end{cases} \\
&\iff \begin{cases} i\sigma(y) = x \\ ix = \sigma(y) \end{cases} \\
&\iff \begin{cases} -x = i^2 x = x \\ ix = \sigma(y) \end{cases} \\
&\iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

and thus $V_\rho^{\mathbb{Q}}$ is the trivial subspace. This shows that the representation ρ is not GC, although it is conjugate over $\mathrm{GL}(n, E)$ to the representation

$$\sigma \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Q})$$

which is GC.

The goal of the remaining part of this section is to show that every rational representation, so every representation into $\mathrm{GL}(m, \mathbb{Q})$, is Galois compatible.

Proposition 9.9. *Let E be a Galois extension of $\mathbb{Q}$. Every representation $\mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{GL}(m, \mathbb{Q})$ is Galois compatible.*

To prove this statement we will first show that it holds for irreducible representations and then apply the following lemma.

Lemma 9.10. *Let $\rho_i : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{GL}(n_i, E)$ with $i \in \{1, 2\}$ be representations, then the following are equivalent:*

1. *ρ_1 and ρ_2 are Galois compatible.*
2. *$\rho_1 \oplus \rho_2 : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{GL}(n_1 + n_2, E)$ is Galois compatible.*

Proof. Let $V_{\rho_1}^{\mathbb{Q}}$ and $V_{\rho_2}^{\mathbb{Q}}$ be the rational subspaces corresponding to ρ_1 and ρ_2, then by definition it follows that

$$V_{\rho_1 \oplus \rho_2}^{\mathbb{Q}} = V_{\rho_1}^{\mathbb{Q}} \oplus V_{\rho_2}^{\mathbb{Q}}.$$

Indeed, we have the following equivalences

$$\begin{aligned}
(v, w) \in V_{\rho_1 \oplus \rho_2}^{\mathbb{Q}} &\iff (v, w)^\sigma = (\rho_1 \oplus \rho_2)_\sigma (v, w) \;\; \forall \sigma \in \mathrm{Gal}(E, \mathbb{Q}) \\
&\iff (v^\sigma, w^\sigma) = ((\rho_1)_\sigma (v), (\rho_2)_\sigma (w)) \;\; \forall \sigma \in \mathrm{Gal}(E, \mathbb{Q}) \\
&\iff v \in V_{\rho_1}^{\mathbb{Q}}, \; w \in V_{\rho_2}^{\mathbb{Q}}
\end{aligned}$$

and thus the claim holds. The statement of the lemma then easily follows by using Lemma 9.4. □

Example 9.8 showed that being *GC* is not preserved under E-equivalence for general fields E. For the field $\mathbb{Q}$ this is true though.

Lemma 9.11. *Let $\rho_i : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{GL}(n, E)$ with $i \in \{1, 2\}$ be representations which are $\mathbb{Q}$-equivalent. Then ρ_1 is GC if and only if ρ_2 is GC.*

Proof. Let $P \in \mathrm{GL}(n, \mathbb{Q})$ such that $\rho_1(\sigma) = P\rho_2(\sigma)P^{-1}$ for every $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$. Then the vector spaces $V_{\rho_1}^{\mathbb{Q}}$ and $V_{\rho_2}^{\mathbb{Q}}$ satisfy

$$V_{\rho_1}^{\mathbb{Q}} = P V_{\rho_2}^{\mathbb{Q}} P^{-1}$$

and thus the statement follows. □

The proof of the proposition is now immediate by using the previous lemmas.

Proof of Proposition 9.9. From Lemma 9.10 and Lemma 9.11, it follows that it is sufficient to prove the proposition for the $\mathbb{Q}$-irreducible representations of $\mathrm{Gal}(E, \mathbb{Q})$. Since every $\mathbb{Q}$-irreducible representation is a subrepresentation of the regular representation (see e.g. [68, Corollary 9.5.]), the statement follows from Example 9.6 above. □

9.2.2 Constructing a rational form of a Lie algebra

In the previous subsection we constructed rational forms of vector spaces E^m from representations of the Galois group $\mathrm{Gal}(E, \mathbb{Q})$. In this subsection we use this technique to construct Lie algebras starting from a representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ into the automorphisms of a rational Lie algebra.

Let $W^{\mathbb{Q}}$ be a rational vector space of dimension n and E a Galois extension of the rational numbers $\mathbb{Q}$. By fixing a basis for $W^{\mathbb{Q}}$, this vector space is isomorphic to $\mathbb{Q}^n$ and thus we have the action of $\mathrm{Gal}(E, \mathbb{Q})$ on W^E as explained above in Section 9.2.1. We claim that this action does not depend on the choice of basis for $W^{\mathbb{Q}}$.

Indeed, take two different bases $v_1, v_2, \ldots, v_n$ and $w_1, \ldots, w_n$ for $W^{\mathbb{Q}}$ and consider the action of $\mathrm{Gal}(E, \mathbb{Q})$ on W^E induced by the basis $v_1, \ldots, v_n$. Because of equations (9.1), it suffices to check that the action corresponds on the basis $w_1, \ldots, w_n$ of the vector space $W^{\mathbb{Q}}$. Now, for every $1 \leq i \leq n$, we can find rational numbers $q_{ij} \in \mathbb{Q}$ such that $w_i = \sum_{j=1}^{n} q_{ij} v_j$. Therefore we have that

$$w_i^\sigma = \sum_{j=1}^{n} \sigma^{-1}(q_{ij}) v_j^\sigma = \sum_{j=1}^{n} q_{ij} v_j = w_i$$

because of equations (9.1). We conclude that the action is independent of the choice of basis and thus is well-defined for every rational vector space.

Now, let $\mathfrak{n}^{\mathbb{Q}}$ be a rational Lie algebra and E a Galois extension of $\mathbb{Q}$ with corresponding action of $\mathrm{Gal}(E, \mathbb{Q})$ on $\mathfrak{n}^{\mathbb{Q}}$. This action satisfies the property

$$[X, Y]^\sigma = [X^\sigma, Y^\sigma]$$

for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$, $X, Y \in \mathfrak{n}^E$. Again, it suffices to check this on a basis of $\mathfrak{n}^E$ as a vector space. By taking a basis for $\mathfrak{n}^E$ such that every basis vector lies in $\mathfrak{n}^{\mathbb{Q}}$, the relation follows immediately.

Let

$$\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}}) < \mathrm{GL}(\mathfrak{n}^{\mathbb{Q}})$$

be any representation, where $\mathrm{GL}(\mathfrak{n}^{\mathbb{Q}})$ is the group of isomorphisms of $\mathfrak{n}^{\mathbb{Q}}$ as vector space. Take $X, Y \in V_\rho^{\mathbb{Q}}$, where $V_\rho^{\mathbb{Q}}$ is again the subspace defined by equation (9.2). Then

$$[X,Y]^\sigma = [X^\sigma, Y^\sigma] = [\rho_\sigma(X), \rho_\sigma(Y)] = \rho_\sigma([X,Y])$$

and thus $[X,Y] \in V_\rho^{\mathbb{Q}}$. We deduce that the rational subspace $V_\rho^{\mathbb{Q}}$ forms a subalgebra of $\mathfrak{n}^E$ and therefore is a rational form of $\mathfrak{n}^E$ by Proposition 9.9.

Proposition 9.12. *Let E be a Galois extension of $\mathbb{Q}$ and $\rho : \mathrm{Gal}(E,\mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ a representation where $\mathfrak{n}^{\mathbb{Q}}$ is a rational Lie algebra. Then the subspace*

$$\mathfrak{m}_\rho^{\mathbb{Q}} = \{v \in \mathfrak{n}^E \mid \forall \sigma \in \mathrm{Gal}(E,\mathbb{Q}), \rho_\sigma(v) = v^\sigma\}$$

is a rational form of the Lie algebra $\mathfrak{n}^E$.

We denote this subspace by $\mathfrak{m}_\rho^{\mathbb{Q}}$ to emphasize the fact that it is a subalgebra. In the next section, we will show how to construct automorphisms on this rational form of the Lie algebra $\mathfrak{n}^E$.

9.2.3 Constructing automorphisms on a rational form

We now answer the question when a given automorphism $f : \mathfrak{n}^E \to \mathfrak{n}^E$ induces an automorphism on $\mathfrak{m}_\rho^{\mathbb{Q}}$. Note that $\mathrm{Gal}(E,\mathbb{Q})$ also acts on the right on $\mathrm{Aut}(\mathfrak{n}^E)$, by defining for all $f \in \mathrm{Aut}(\mathfrak{n}^E)$ the action as

$$f^\sigma(v) = \left(f\left(v^{\sigma^{-1}}\right)\right)^\sigma.$$

By fixing a basis for $\mathfrak{n}^{\mathbb{Q}}$ and thus also for $\mathfrak{n}^E$, the matrix representation of f^σ is given by applying σ^{-1} to every entry of the matrix representation of f. Also every representation $\rho : \mathrm{Gal}(E,\mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^E)$ induces a left action on $\mathrm{Aut}(\mathfrak{n}^E)$ by conjugation. The automorphisms where this left action corresponds with the right action are exactly those that induce an automorphism on $\mathfrak{m}_\rho^{\mathbb{Q}}$:

Lemma 9.13. *Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational Lie algebra and $\rho : G \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ a representation. An element $f \in \mathrm{Aut}(\mathfrak{n}^E)$ induces an automorphism on $\mathfrak{m}_\rho^{\mathbb{Q}}$ if and only if $f^\sigma = \rho_\sigma f \rho_{\sigma^{-1}}$ for all $\sigma \in \mathrm{Gal}(E,\mathbb{Q})$.*

Proof. First assume that f satisfies the condition, i.e. that $f^\sigma = \rho_\sigma f \rho_{\sigma^{-1}}$ for $\sigma \in \mathrm{Gal}(E,\mathbb{Q})$. For every $v \in \mathfrak{m}_\rho^{\mathbb{Q}}$ and $\sigma \in \mathrm{Gal}(E,\mathbb{Q})$, we have that

$$\big(f(v)\big)^\sigma = f^\sigma(v^\sigma) = \rho_\sigma f \rho_{\sigma^{-1}}(\rho_\sigma(v)) = \rho_\sigma(f(v))$$

and thus $f(v) \in \mathfrak{m}_\rho^\mathbb{Q}$ because σ was taken arbitrary. Since $\mathfrak{m}_\rho^\mathbb{Q}$ is a rational form, the restriction of f to $\mathfrak{m}_\rho^\mathbb{Q}$ is invertible and f induces an automorphism of $\mathfrak{m}_\rho^\mathbb{Q}$.

For the other direction, fix $\sigma \in \operatorname{Gal}(E, \mathbb{Q})$ and fix a basis $\{v_1, \ldots v_m\} \subseteq \mathfrak{m}_\rho^\mathbb{Q}$ for the Lie algebra $\mathfrak{n}^E$. Since f induces an automorphism on $\mathfrak{m}_\rho^\mathbb{Q}$ we know that $f(v_i) \in \mathfrak{m}_\rho^\mathbb{Q}$ and thus that $\big(f(v_i)\big)^\sigma = \rho_\sigma(f(v_i))$. Since the vectors $\rho_\sigma(v_i)$ also form a basis for the Lie algebra $\mathfrak{n}^E$, it suffices to prove the relation $f^\sigma = \rho_\sigma f \rho_{\sigma^{-1}}$ on this basis. A computation shows that

$$\rho_\sigma f \rho_{\sigma^{-1}}(\rho_\sigma(v_i)) = \rho_\sigma(f(v_i)) = \big(f(v_i)\big)^\sigma = f^\sigma(v_i^\sigma) = f^\sigma(\rho_\sigma(v_i))$$

and thus the relation holds. $\square$

The main theorem of this chapter is the combination of Proposition 9.12 and Lemma 9.13.

Theorem 9.14. *Let $\mathfrak{n}^\mathbb{Q}$ be a rational Lie algebra and $\rho : \operatorname{Gal}(E, \mathbb{Q}) \to \operatorname{Aut}(\mathfrak{n}^\mathbb{Q})$ a representation. Suppose there exists a Lie algebra automorphism $f : \mathfrak{n}^E \to \mathfrak{n}^E$ such that $\rho_\sigma f \rho_{\sigma^{-1}} = f^\sigma$ for all $\sigma \in \operatorname{Gal}(E, \mathbb{Q})$. Then there also exists a rational form $\mathfrak{m}^\mathbb{Q} \subseteq \mathfrak{n}^E$ such that f induces an automorphism of $\mathfrak{m}^\mathbb{Q}$.*

If all eigenvalues of f are algebraic units of absolute value different from 1, then $\mathfrak{m}^\mathbb{Q}$ is Anosov.

Although the formulation of this result is rather technical, we will illustrate in Section 9.3 how it has many applications for the existence of Anosov diffeomorphisms on nilmanifolds. In Corollary 9.15 we will also give a more practical version of this result.

Proof. The only statement left to show is the last one. We claim that if f is an automorphism of a rational Lie algebra with only algebraic units as eigenvalues, then f is integer-like. Note that the coefficients of the characteristic polynomial of f are formed by taking sums and products of the eigenvalues and thus all these coefficients are algebraic integers. Since these coefficients also lie in $\mathbb{Q}$, they are integers. The determinant of f is equal to the product of all eigenvalues and therefore is an algebraic unit. Since the only algebraic units in $\mathbb{Q}$ are ± 1, the claim now follows and this ends the proof. $\square$

Note that the type of the Lie algebra $\mathfrak{m}^\mathbb{Q}$ of the theorem is equal to the type of $\mathfrak{n}^\mathbb{Q}$ and is thus completely determined. The signature of the automorphism f also does not change by restricting it to a rational form. But in general the Lie algebras $\mathfrak{n}^\mathbb{Q}$ and $\mathfrak{m}^\mathbb{Q}$ will not be isomorphic, so this theorem does not allow us to show that a specific Lie algebra is Anosov. In Section 9.3.2 we determine the isomorphism class of the Lie algebra in a low-dimensional case by using the classification of low-dimensional Anosov algebras given in [75]. In Chapter 11, we state the question whether the techniques of this section can be generalized to determine which of the Lie algebras constructed by Theorem 9.14 are isomorphic.

In most examples the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ is given by the eigenspaces of an automorphism. In this special case, Theorem 9.14 becomes easier to handle.

Corollary 9.15. Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational Lie algebra and assume there exists a decomposition of $\mathfrak{n}^{\mathbb{Q}}$ into subspaces

$$\mathfrak{n}^{\mathbb{Q}} = \bigoplus_{\lambda \in E} V_\lambda$$

such that $[V_\lambda, V_\mu] \subseteq V_{\lambda\mu}$. Let $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ be a representation such that $\rho_\sigma(V_\lambda) = V_{\sigma(\lambda)}$ for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$. Then the linear map $f : \mathfrak{n}^E \to \mathfrak{n}^E$ given by $f(X) = \lambda X$ for all $X \in V_\lambda$ induces an automorphism on some rational form $\mathfrak{m}^{\mathbb{Q}} \subseteq \mathfrak{n}^E$.

If every λ is an algebraic unit of absolute value different from 1, then $\mathfrak{m}^{\mathbb{Q}}$ is Anosov.

Proof. We will make use of Theorem 9.14 to prove this. The condition on the Lie bracket implies that f is a Lie algebra automorphism and the last condition is identical to the last condition of Theorem 9.14. So it is left to show that

$$\rho_\sigma f \rho_{\sigma^{-1}} = f^\sigma$$

for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$. It suffices to prove this relation for vectors $X \in V_\lambda$ for all possible λ. For such a vector, we have that

$$\rho_\sigma f \rho_{\sigma^{-1}}(X) = \rho_\sigma\Big(f\big(\rho_{\sigma^{-1}}(X)\big)\Big) = \sigma^{-1}(\lambda)\rho_\sigma\big(\rho_{\sigma^{-1}}(X)\big) = \sigma^{-1}(\lambda)\, X$$

since $\rho_{\sigma^{-1}}(X) \in V_{\sigma^{-1}(\lambda)}$ and

$$f^\sigma(X) = (f(X))^\sigma = (\lambda X)^\sigma = \sigma^{-1}(\lambda) X$$

since $X^{\sigma^{-1}} = X$ and thus equality holds. ☐

Remark 9.16. Corollary 9.15 is the version of the main theorem we will use most of the times. In many examples, the spaces V_λ will be one-dimensional and thus given by basis vectors X_λ for $\lambda \in E$. In these examples, the Lie bracket is of the form

$$[X_\lambda, X_\mu] = \pm X_{\lambda\mu}$$

and the representation ρ is given by $\rho_\sigma(X_\lambda) = \pm X_{\sigma(\lambda)}$ for all $\lambda \in E, \sigma \in \mathrm{Gal}(E, \mathbb{Q})$.

It still has to be checked that the map ρ indeed defines a representation, i.e. that the signs in the definition can be chosen such that $\rho_\sigma \in \mathrm{Aut}(\mathfrak{n}^\mathbb{Q})$ and $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^\mathbb{Q})$ is a group morphism. If we know that ρ is a representation, then we can use Corollary 9.15 on the subspaces V_λ spanned by X_λ to conclude that $\mathfrak{n}^E$ has a rational form which is Anosov.

At this point, we want to emphasize the connection between Corollary 9.15 and Chapter 8. The decomposition of this corollary is indeed a grading of the Lie algebra $\mathfrak{n}^\mathbb{Q}$, but indexed by the group U_E of algebraic units of the field E. The statement of Corollary 9.15 is then that the existence of a U_E-grading which is invariant under the action of $\mathrm{Gal}(E, \mathbb{Q})$ implies that the Lie algebra $\mathfrak{n}^E$ has a rational form which is Anosov.

9.3 Applications of the construction

The main application of the theorem and corollary above lies in constructing Anosov Lie algebras of specific types and with Anosov automorphisms of specific signatures. We present here five different consequences of the main theorem, going from simplifying and correcting existing results to constructing new examples of minimal signature and minimal type and also the construction of a nilmanifold admitting an Anosov diffeomorphism but no expanding map.

9.3.1 Simplification of existing results

By using our main theorem, all the examples of Anosov Lie algebras in [72, 74, 75, 81] are now straightforward to construct. For instance, the examples in [75, 81] follow from Remark 9.16 since they start from a basis for the one-dimensional eigenspaces of a hyperbolic automorphism. As another example of the simplification we demonstrate how [74, Theorem 3.2.] follows after a few lines.

Let $\mathfrak{n}^E$ be a Lie algebra over the field E and assume there exists a positive grading

$$\mathfrak{n}^E = \mathfrak{n}_1^E \oplus \mathfrak{n}_2^E \oplus \ldots \oplus \mathfrak{n}_k^E$$

of $\mathfrak{n}^E$. For every $\lambda \in E$, let $f_\lambda : \mathfrak{n}^E \to \mathfrak{n}^E$ be the automorphism given by

$$f_\lambda(X) = \lambda^i X \quad \forall X \in \mathfrak{n}_i.$$

Just as in Chapter 8 the definition of a grading implies that f_λ is indeed an automorphism of the Lie algebra. The proof of [74, Theorem 3.2.] now follows immediately from the main theorem.

Theorem 9.17. *Let $\mathfrak{n}^{\mathbb{Q}}$ be a Lie algebra with a positive grading and consider the direct sum*

$$\tilde{\mathfrak{n}}^{\mathbb{Q}} = \underbrace{\mathfrak{n}^{\mathbb{Q}} \oplus \ldots \oplus \mathfrak{n}^{\mathbb{Q}}}_{m \text{ times}}$$

with $m \geq 2$. Then there exists a rational form of $\tilde{\mathfrak{n}}^{\mathbb{R}}$ which is Anosov.

Proof. Let E be any real Galois extension of $\mathbb{Q}$ of degree m with $\mathrm{Gal}(E, \mathbb{Q}) = \{\sigma_1, \ldots, \sigma_m\}$. Every $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ induces a permutation $\pi \in S_m$ via

$$\sigma\sigma_i = \sigma_{\pi(i)} \text{ for all } i \in \{1, \ldots, m\},$$

and thus also an automorphism $\rho_\sigma \in \mathrm{Aut}(\tilde{\mathfrak{n}}^{\mathbb{Q}})$ by permuting the components of $\tilde{\mathfrak{n}}^{\mathbb{Q}}$ according to π. Note that ρ is a representation $\mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\tilde{\mathfrak{n}}^{\mathbb{Q}})$.

From Chapter 4, it follows that there exists unit Pisot numbers in E because $m \geq 2$. Fix a unit Pisot number λ in E and take the grading $\mathfrak{n}^{\mathbb{Q}} = \mathfrak{n}_1^{\mathbb{Q}} \oplus \cdots \oplus \mathfrak{n}_k^{\mathbb{Q}}$ for $\mathfrak{n}^{\mathbb{Q}}$. By writing the subspace $\mathfrak{n}_i^{\mathbb{Q}}$ of the j-th component of $\tilde{\mathfrak{n}}^{\mathbb{Q}}$ as $V_{\sigma_i(\lambda^j)}$, it is immediate from the construction of ρ that all conditions of Corollary 9.15 are satisfied. Since every $\sigma_i(\lambda^j)$ is an algebraic unit of absolute value different from 1, it follows that $\tilde{\mathfrak{n}}^E$ (and therefore also $\tilde{\mathfrak{n}}^{\mathbb{R}}$) has a rational form which is Anosov. $\square$

9.3.2 Correction to a result of Lauret and Will

In [75], a classification of all Anosov Lie algebras up to dimension 8 is given. As one of the consequences, as we recalled in (False) Claim 9.1, it is stated in [75, Corollary 4.3.] that every Anosov diffeomorphism on a nilmanifold of dimension ≤ 8 which is not a torus, has signature $\{3, 3\}$ or $\{4, 4\}$. There is not really a proof of this statement given in [75] and in fact, by using Theorem 9.14 we can give new examples which were overlooked by the author.

Pfaffian form

First we recall some notions of [75] about the Pfaffian form of a Lie algebra. Let $\mathfrak{n}^E$ be a Lie algebra over any field E and take $\langle\ ,\ \rangle$ an inner product on $\mathfrak{n}^E$, i.e. a non-degenerate symmetric bilinear form. For every $Z \in \mathfrak{n}^E$, there exists a linear map $J_Z : \mathfrak{n}^E \to \mathfrak{n}^E$ defined by

$$\langle J_Z X, Y\rangle = \langle [X,Y], Z\rangle \quad \forall X, Y \in \mathfrak{n}^E.$$

Note that J_Z is skew-symmetric with respect to the inner product, meaning that

$$\langle J_Z X, Y\rangle = -\langle X, J_Z Y\rangle$$

for all $X, Y \in \mathfrak{n}^E$. It is easy to check that an isomorphism $\alpha : \mathfrak{n}^E \to \mathfrak{n}^E$ is an automorphism of $\mathfrak{n}^E$ if and only if

$$\alpha^T J_Z \alpha = J_{\alpha^T(Z)} \tag{9.4}$$

for all $Z \in \mathfrak{n}^E$, where $\alpha^T : \mathfrak{n}^E \to \mathfrak{n}^E$ is the adjoint map of α, defined by $\langle X, \alpha^T(Y)\rangle = \langle \alpha(X), Y\rangle$.

Now assume that $\mathfrak{n}^E$ is a 2-step nilpotent Lie algebra of type $(2m, k)$ and take $V^E \subseteq \mathfrak{n}^E$ such that $V^E \oplus \gamma_2(\mathfrak{n}^E) = \mathfrak{n}^E$ as a vector space. Take an inner product satisfying $\langle V^E, \gamma_2(\mathfrak{n}^E)\rangle = 0$ and such that there exists an orthonormal basis for V^E with respect to the inner product.

Denote the vector space (in fact it is a Lie algebra) of all skew-symmetric endomorphisms of V^E by $\mathfrak{so}(V^E)$. The construction above then induces a linear map

$$J : \gamma_2(\mathfrak{n}^E) \to \mathfrak{so}(V^E)$$

$$Z \mapsto J_Z|_{V^E}.$$

After taking an orthonormal basis for V^E, we can identify $\mathfrak{so}(V^E)$ with the skew-symmetric matrices. Recall that the Pfaffian on V^E is the unique polynomial function

$$\mathrm{Pf} : \mathfrak{so}(V^E) \to E$$

such that $\mathrm{Pf}(A)^2 = \det(A)$ for all $A \in \mathfrak{so}(V^E)$ and $\mathrm{Pf}(S) = 1$ for some fixed $S \in \mathfrak{so}(V^E)$ with $\det(S) = 1$ (where this last condition is needed to fix the sign). For every endomorphism $A : V^E \to V^E$, we have the relation

$$\det(A^T B A) = \det(A)^2 \det(B).$$

This implies that $\mathrm{Pf}(A^T B A) = \pm \det(A)\,\mathrm{Pf}(B)$ and it is an exercise to check that the sign $\pm$ does not depend on the matrix A. By taking $A = \mathbb{1}_{V^E}$ we conclude that the Pfaffian satisfies the relation $\mathrm{Pf}(A^T B A) = \det(A)\,\mathrm{Pf}(B)$ for all endomorphisms $A : V^E \to V^E$.

The composition
$$h = \mathrm{Pf} \circ J : \gamma_2(\mathfrak{n}^E) \to E$$
is called the Pfaffian form of the 2-step nilpotent Lie algebra $\mathfrak{n}^E$. By taking another vector space V^E or changing the inner product on $\mathfrak{n}^E$ or the basis of V^E, the Pfaffian form changes to a polynomial

$$\begin{aligned} \gamma_2(\mathfrak{n}^E) &\to E \\ Z &\mapsto eh(\beta(Z)) \end{aligned}$$

for some $e \in E^\times$ and some isomorphism $\beta : \gamma_2(\mathfrak{n}^E) \to \gamma_2(\mathfrak{n}^E)$. Thus the Pfaffian form is uniquely determined by $\mathfrak{n}^E$ up to projective equivalence (see [73, Proposition 2.4.] for the exact definition and proof of this statement). The polynomial h is homogeneous of degree m in k variables and with coefficients in E. An automorphism of the Pfaffian form is an isomorphism $\beta : \gamma_2(\mathfrak{n}^E) \to \gamma_2(\mathfrak{n}^E)$ such that $h \circ \beta = h$.

A binary quadratic form over $\mathbb{Q}$ is a homogeneous polynomial of degree 2 in 2 variables, i.e. polynomials that can be written as

$$h(X, Y) = aX^2 + bXY + cY^2$$

with $a, b, c \in \mathbb{Q}$. The Pfaffian form of a rational Lie algebra of type $(4, 2)$ is such a polynomial. The discriminant $\Delta(h)$ of h is defined as

$$\Delta(h) = b^2 - 4ac.$$

In [56] it is shown that two rational Lie algebras of type $(4, 2)$ with Pfaffian forms h_1 and h_2 are isomorphic if and only if there exists $q \in \mathbb{Q}^\times$ such that $\Delta(h_1) = q^2 \Delta(h_2)$. Given any $k \in \mathbb{Z}$, we can consider the Lie algebra $\mathfrak{n}_k^{\mathbb{Q}}$ given by the basis $X_1, X_2, X_3, X_4, Z_1, Z_2$ and relations

$$\begin{aligned} [X_1, X_3] &= Z_1 & \qquad [X_1, X_4] &= Z_2 \\ [X_2, X_3] &= kZ_2 & \qquad [X_2, X_4] &= Z_1. \end{aligned}$$

The Lie algebra $\mathfrak{n}_k^{\mathbb{Q}}$ has Pfaffian form equal to $h(X, Y) = X^2 - kY^2$ with discriminant $4k$. So every rational Lie algebra of type $(4, 2)$ is isomorphic to $\mathfrak{n}_k^{\mathbb{Q}}$ for some $k \in \mathbb{Z}$ and $\mathfrak{n}_k^{\mathbb{Q}}$ is isomorphic to $\mathfrak{n}_{k'}^{\mathbb{Q}}$ if and only if there exists a natural number $q > 0$ such that $k = q^2 k'$.

The automorphisms $\beta \in \mathrm{SL}(2, \mathbb{Z})$ of a binary quadratic form $h(X, Y) = aX^2 + bXY + cY^2$ are completely known and described e.g. in [18, Theorem 2.5.5.]. Given a solution $x, y \in \mathbb{Z}$ of the Pell equation

$$x^2 - \Delta(h)y^2 = 4,$$

the matrix

$$U(x,y) = \begin{pmatrix} \frac{x-yb}{2} & -cy \\ ay & \frac{x+yb}{2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

is an automorphism of the quadratic form h. The map U is a bijection between the solutions of the Pell equation and the automorphisms of h which lie in $\mathrm{SL}(2,\mathbb{Z})$. The eigenvalues of $U(x,y)$ are equal to $\frac{x\pm\sqrt{\Delta(h)}y}{2}$, so the field in which these eigenvalues lie gives us information about the discriminant of the form h.

Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational Lie algebra of type $(4,2)$ and $\alpha \in \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ an Anosov automorphism. By squaring α if necessary, we can also assume that $\det(\alpha) = 1$. From the equation $\alpha^T J_Z \alpha = J_{\alpha^T(Z)}$ (see (9.4)) and by applying the Pfaffian, we get that $h(\alpha^T(Z)) = h(Z)$ for all $Z \in V^{\mathbb{Q}}$. So α induces a hyperbolic and integer-like automorphism $\beta = \alpha^T|_{\gamma_2(\mathfrak{n}^{\mathbb{Q}})}$ of the Pfaffian form h. The eigenvalues of β (and thus also of $\alpha|_{\gamma_2(\mathfrak{n}^{\mathbb{Q}})}$) lie in the field $\mathbb{Q}(\sqrt{\Delta(h)})$ and thus if we know these eigenvalues, we can determine the discriminant of the Pfaffian form of $\mathfrak{n}^{\mathbb{Q}}$ up to a square and therefore also the isomorphism class of $\mathfrak{n}^{\mathbb{Q}}$. This also implies that every Anosov Lie algebra of type $(4,2)$ is isomorphic to a Lie algebra $\mathfrak{n}_k^{\mathbb{Q}}$ with k a square free natural number > 1 (since all other values of k imply that the eigenvalues have absolute value 1). This gives a simplified proof of the result that the only non-abelian Anosov Lie algebras of dimension 6 are isomorphic to $\mathfrak{n}_k^{\mathbb{Q}}$ with k a square free natural number > 1, see [82] and [74].

Construction

We now have all the tools to construct new Anosov automorphisms on the Anosov Lie algebras $\mathfrak{n}_k^{\mathbb{Q}}$:

Proposition 9.18. *Let k be a natural number with $k > 1$ and k square free. Let $\mathfrak{n}_k^{\mathbb{Q}}$ be the Lie algebra with basis $X_1, X_2, X_3, X_4, Z_1, Z_2$ and relations*

$$[X_1, X_3] = Z_1 \qquad [X_1, X_4] = Z_2$$

$$[X_2, X_3] = kZ_2 \qquad [X_2, X_4] = Z_1.$$

Then there exists an Anosov automorphism f on $\mathfrak{n}_k^{\mathbb{Q}}$ with $\mathrm{sgn}(f) = \{2,4\}$.

Proof. Fix the k of the theorem and let l be a different natural number with l square free and $l > 1$. Take $E = \mathbb{Q}(\sqrt{k}, \sqrt{l})$, then E is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(E,\mathbb{Q}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $\tau \in \mathrm{Gal}(E,\mathbb{Q})$ be the unique element with $\tau(\sqrt{k}) = \sqrt{k}, \tau(\sqrt{l}) = -\sqrt{l}$ and take another $\sigma \in \mathrm{Gal}(E,\mathbb{Q})$ such that $\mathrm{Gal}(E,\mathbb{Q}) =$

$\{1, \sigma, \tau, \sigma\tau\}$. Take $\lambda_1 \in E$ an unit Pisot number as introduced in the Appendix. Write the Galois conjugates of λ_1 as $\lambda_1, \tau(\lambda_1) = \lambda_2, \sigma(\lambda_1) = \lambda_3, \sigma\tau(\lambda_1) = \lambda_4$

Consider the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ with basis $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_3}, X_{\lambda_4}, Y_{\lambda_1\lambda_2}, Y_{\lambda_3\lambda_4}$ and Lie bracket given by

$$[X_{\lambda_1}, X_{\lambda_2}] = Y_{\lambda_1\lambda_2}$$

$$[X_{\lambda_3}, X_{\lambda_4}] = Y_{\lambda_3\lambda_4}$$

and all other brackets zero (so $\mathfrak{n}^{\mathbb{Q}}$ is isomorphic to the direct sum of two copies of the Heisenberg Lie algebra of dimension 3). Each of these basis vectors spans a 1-dimensional subspace indexed by the same algebraic unit, corresponding to the decomposition in Corollary 9.15. Consider the representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ induced by $\rho_\sigma(X_\lambda) = X_{\sigma(\lambda)}$ and $\rho_\tau(X_\lambda) = X_{\tau(\lambda)}$. A small computation shows that this is indeed a representation. By using Corollary 9.15 we then get a rational form $\mathfrak{m}^{\mathbb{Q}}$ of $\mathfrak{n}^E$ with Anosov automorphism $f : \mathfrak{m}^{\mathbb{Q}} \to \mathfrak{m}^{\mathbb{Q}}$.

Note that f has only two eigenvalues > 1, namely λ_1 and $\lambda_1\lambda_2$ and thus $\mathrm{sgn}(f) = \{2, 4\}$. Since $\tau(\lambda_1\lambda_2) = \lambda_1\lambda_2$ and $\mathbb{Q}(\sqrt{k})$ is the unique subfield of E fixed by τ, the eigenvalue $\lambda_1\lambda_2$ is in $\mathbb{Q}(\sqrt{k})$, showing that the Pfaffian form h of $\mathfrak{m}^{\mathbb{Q}}$ satisfies $\Delta(h) = q^2 k$ for some $q \in \mathbb{Q}$. This shows that $\mathfrak{m}^{\mathbb{Q}}$ is isomorphic to $\mathfrak{n}_k^{\mathbb{Q}}$ and thus $\mathfrak{n}_k^{\mathbb{Q}}$ also has an Anosov automorphism of signature $\{2, 4\}$. $\square$

It is also possible to start the proof from a Galois extension $\mathbb{Q} \subseteq E$ with $\mathrm{Gal}(E, \mathbb{Q}) = \mathbb{Z}_4$ as we show in the example below. By computing a basis for $\mathfrak{m}^{\mathbb{Q}}$ we give an explicit example which was overlooked in (False) Claim 9.1:

Example 9.19. Start from the polynomial

$$p(X) = X^4 - 4X^3 - 4X^2 + X + 1,$$

which has 4 distinct real roots, say $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. These roots satisfy $\lambda_1 > 1$ and $|\lambda_i| < 1$ for $i \in \{2, 3, 4\}$, showing that λ_1 is an unit Pisot number. The Galois group of the field $E = \mathbb{Q}(\lambda_1)$ is isomorphic to $\mathbb{Z}_4$, which can be checked e.g. with GAP, see [52]. A generator $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ is given by

$$\sigma(\lambda_1) = \lambda_2,\ \sigma(\lambda_2) = \lambda_3,\ \sigma(\lambda_3) = \lambda_4,\ \sigma(\lambda_4) = \lambda_1.$$

Just as in the proof of Proposition 9.18, consider the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ with basis

$$X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_3}, X_{\lambda_4}, Y_{\lambda_1\lambda_3}, Y_{\lambda_2\lambda_4}$$

and Lie bracket given by

$$[X_{\lambda_1}, X_{\lambda_3}] = Y_{\lambda_1\lambda_3}$$

$$[X_{\lambda_2}, X_{\lambda_4}] = Y_{\lambda_2\lambda_4}$$

Consider the representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ induced by $\rho_\sigma(X_{\lambda_i}) = X_{\sigma(\lambda_i)}$. The main theorem guarantees us the existence of a rational form $\mathfrak{m}^{\mathbb{Q}}$ of $\mathfrak{n}^E$ which is Anosov, but we now compute this Lie algebra explicitly by giving a basis for $\mathfrak{m}^{\mathbb{Q}}$.

Consider the basis $U_1, U_2, U_3, U_4, V_1, V_2$ given by

$$\begin{aligned} U_i &= \sum_{j=1}^{4} \lambda_j^{i-1} X_{\lambda_j} \\ V_i &= \left(\lambda_3^i - \lambda_1^i\right) Y_{\lambda_1\lambda_3} + \left(\lambda_4^i - \lambda_2^i\right) Y_{\lambda_2\lambda_4}. \end{aligned}$$

To simplify the computations, we will use the notations $\lambda_0 = \lambda_4$ and $\lambda_5 = \lambda_1$. The basis vectors U_i satisfy

$$U_i^\sigma = \sum_{j=1}^{4} \left(\lambda_j^{i-1}\right)^\sigma X_{\lambda_j} = \sum_{j=1}^{4} \sigma^{-1}\left(\lambda_j^{i-1}\right) X_{\lambda_j} = \sum_{j=1}^{4} \lambda_{j-1}^{i-1} X_{\lambda_j}$$

and

$$\rho_\sigma(U_i) = \sum_{j=1}^{4} \lambda_j^{i-1} \rho_\sigma\left(X_{\lambda_j}\right) = \sum_{j=1}^{4} \lambda_j^{i-1} X_{\lambda_{j+1}} = \sum_{j=2}^{5} \lambda_{j-1}^{i-1} X_{\lambda_j}.$$

We conclude that $\rho_\sigma(U_i) = U_i^\sigma$ and similarly this equation also holds for the vectors V_i. This shows that the basis vectors $U_1, U_2, U_3, U_4, V_1, V_2$ satisfy the defining relation of the rational form given in equation (9.2) of Section 9.2.1 and thus they indeed span the rational form $\mathfrak{m}^{\mathbb{Q}}$ of Theorem 9.14. The induced Anosov automorphism on $\mathfrak{m}^{\mathbb{Q}}$ guaranteed by Theorem 9.14 is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{5}{2} \end{pmatrix}$$

in the basis $U_1, U_2, U_3, U_4, V_1, V_2$. By using the matrix representation of this Anosov automorphism, one can compute the Lie bracket:

$$\begin{aligned} &[U_1, U_2] = V_1 && [U_1, U_3] = V_2 \\ &[U_2, U_3] = -\frac{1}{2} V_1 - \frac{1}{2} V_2 && [U_2, U_4] = -\frac{1}{2} V_1 - \frac{5}{2} V_2 \\ &[U_3, U_4] = \frac{1}{2} V_1 + \frac{3}{2} V_2 && [U_1, U_4] = \frac{3}{2} V_1 + \frac{9}{2} V_2. \end{aligned}$$

The discriminant of the Pfaffian form of $\mathfrak{m}^{\mathbb{Q}}$ is $\frac{5}{4}$, so $\mathfrak{m}^{\mathbb{Q}}$ is isomorphic to $\mathfrak{n}_5^{\mathbb{Q}}$. The characteristic polynomial of the Anosov automorphism restricted to $\gamma_2(\mathfrak{m}^{\mathbb{Q}})$ is equal to $X^2 + 3X + 1$ which has $\mathbb{Q}(\sqrt{5})$ as splitting field.

Scheuneman duality

Of course, Proposition 9.18 also gives us examples of Anosov automorphisms $f : \mathfrak{n}_k^{\mathbb{Q}} \oplus \mathbb{Q}^2 \to \mathfrak{n}_k^{\mathbb{Q}} \oplus \mathbb{Q}^2$ with $\operatorname{sgn}(f) = \{3, 5\}$. But by using the notion of Scheuneman duality (see [92]) for 2-step nilpotent Lie algebras, it is possible to give another class of Anosov automorphisms overlooked in (False) Claim 9.1. First we recall some details about this method as described in [73].

Let $V^{\mathbb{Q}}$ be any vector space with an inner product and consider the standard inner product B on $\mathfrak{so}(V^{\mathbb{Q}})$, given by

$$B(Z_1, Z_2) = \operatorname{Tr}(Z_1^T Z_2) = -\operatorname{Tr}(Z_1 Z_2).$$

For every subspace $W^{\mathbb{Q}}$ of $\mathfrak{so}(V^{\mathbb{Q}})$, there exists a rational Lie algebra $\mathfrak{n}^{\mathbb{Q}} = V^{\mathbb{Q}} \oplus W^{\mathbb{Q}}$ with Lie bracket $[\,,\,] : V^{\mathbb{Q}} \times V^{\mathbb{Q}} \to W^{\mathbb{Q}}$ defined by

$$B([X, Y], Z) = \langle Z(X), Y\rangle$$

for all $Z \in \mathfrak{so}(V^{\mathbb{Q}})$, $X, Y \in V^{\mathbb{Q}}$. This is a 2-step nilpotent Lie algebra where the map $J : \gamma_2(V^{\mathbb{Q}} \oplus W^{\mathbb{Q}}) = W^{\mathbb{Q}} \to \mathfrak{so}(V^{\mathbb{Q}})$ is the inclusion. If we take the vector space $W^{\mathbb{Q}} = \mathfrak{so}(V^{\mathbb{Q}})$, then the result is the free 2-step nilpotent Lie algebra on $V^{\mathbb{Q}}$. An isomorphism $\alpha : V^{\mathbb{Q}} \to V^{\mathbb{Q}}$ induces an automorphism $\bar{\alpha} : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ if and only if $\alpha^T Z \alpha \in W^{\mathbb{Q}}$ for all $Z \in W^{\mathbb{Q}}$.

Example 9.20. Let $X = (V, E)$ be a graph, then the complement or inverse graph of X is defined as

$$X^c = (V, K \setminus E).$$

Denote by $\mathfrak{g}_X$ the Lie algebra associated to this graph, as explained in the Appendix. The dual of the Lie algebra $\mathfrak{g}_X$ is exactly the Lie algebra $\mathfrak{g}_{X^c}$.

Let $\mathfrak{n}^{\mathbb{Q}}$ be a Lie algebra of type (m, k) and consider the map $J : \gamma_2(\mathfrak{n}^{\mathbb{Q}}) \to \mathfrak{so}(V^{\mathbb{Q}})$ as introduced above. Denote the image of J as $W^{\mathbb{Q}}$, then it follows by definition that $\mathfrak{n}^{\mathbb{Q}}$ is isomorphic to the Lie algebra $V^{\mathbb{Q}} \oplus W^{\mathbb{Q}}$ of the previous paragraph. The dual of $\mathfrak{n}^{\mathbb{Q}}$ is then the Lie algebra $\tilde{n}^{\mathbb{Q}} = V^{\mathbb{Q}} \oplus \tilde{W}^{\mathbb{Q}}$ with Lie bracket as in the previous paragraph, where $\tilde{W}^{\mathbb{Q}}$ is the orthogonal complement of $W^{\mathbb{Q}}$ in $\mathfrak{so}(V^{\mathbb{Q}})$ relative to the inner product B given above. The dual of the Lie algebra $\mathfrak{n}_k^{\mathbb{Q}}$ is denoted by $\mathfrak{h}_k^{\mathbb{Q}}$. If $\alpha \in \mathrm{GL}(V^{\mathbb{Q}})$ induces an automorphism $\bar{\alpha}$ of $\mathfrak{n}^{\mathbb{Q}}$, then α^T induces an automorphism on $\tilde{\mathfrak{n}}^{\mathbb{Q}}$ since $\alpha \tilde{W}^{\mathbb{Q}} \alpha^T = \tilde{W}^{\mathbb{Q}}$ and this map is called the dual automorphism of $\bar{\alpha}$. The combined eigenvalues of $\bar{\alpha}$ and its dual on

$\gamma_2(\mathfrak{n}^{\mathbb{Q}})$ and $\gamma_2(\tilde{\mathfrak{n}}^{\mathbb{Q}})$ are equal to the eigenvalues of the map that α induces on $\gamma_2(V^{\mathbb{Q}} \oplus \mathfrak{so}(V^{\mathbb{Q}}))$ where $V^{\mathbb{Q}} \oplus \mathfrak{so}(V^{\mathbb{Q}})$ is the free 2-step nilpotent Lie algebra on $V^{\mathbb{Q}}$.

The dual Lie algebra of $\mathfrak{n}_k^{\mathbb{Q}}$ is of type $(4,4)$ and denoted as $\mathfrak{h}_k^{\mathbb{Q}}$. The Lie algebra $\mathfrak{h}_k^{\mathbb{Q}}$ can also be described as the one with basis $X_1, X_2, X_3, X_4, Z_1, Z_2, Z_3, Z_4$ and relations

$$\begin{aligned} [X_1, X_2] &= Z_1 & \qquad [X_2, X_3] &= -Z_3 \\ [X_1, X_3] &= Z_2 & \qquad [X_2, X_4] &= -Z_2 \\ [X_1, X_4] &= kZ_3 & \qquad [X_3, X_4] &= Z_4, \end{aligned}$$

see for example [73]. From the Scheuneman duality, the following proposition is immediate:

Proposition 9.21. *For every $k \in \mathbb{N}$ with $k > 1$ and k not a square, there exists an Anosov automorphism on $\mathfrak{h}_k^{\mathbb{Q}}$ with $\operatorname{sgn}(f) = \{3,5\}$. On $\mathfrak{h}_1^{\mathbb{Q}}$ every Anosov automorphism has signature $\{4,4\}$.*

Proof. Note that the dual of an Anosov automorphism with $\operatorname{sgn}(f) = \{2,4\}$ is also Anosov with signature $\{3,5\}$ and vice versa. So the first part follows from the fact that $\mathfrak{h}_k^{\mathbb{Q}}$ is the dual of $\mathfrak{n}_k^{\mathbb{Q}}$. Also the second statement follows since the Lie algebra $\mathfrak{n}_1^{\mathbb{Q}}$ is not Anosov. $\square$

For all other Lie algebras, (False) Claim 9.1 is correct (and the arguments to prove it are the same as the ones used to prove the classification of Anosov Lie algebras up to dimension 8). Also, the Lie algebra $\mathfrak{h}_k^{\mathbb{Q}}$ does not admit an Anosov automorphism of signature $(2,6)$, for example by using the same number theoretical arguments as in [75]. So the combined results above determine completely for which Anosov Lie algebras (False) Claim 9.1 is indeed false.

Note that the examples of Proposition 9.18 also answer Question 9.1 about non-abelian examples of signature $\{2,q\}$ for some $q \in \mathbb{N}_0$. We give a more general approach to this question in the next section.

9.3.3 Anosov automorphisms of minimal signature

In this subsection, we show how the main theorem can be used to construct Anosov automorphisms of minimal signature. These examples answer Question 9.1 which we already mentioned in the introduction.

Let $\mathfrak{n}^{\mathbb{Q}}$ be an Anosov Lie algebra of nilpotency class c and $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ a hyperbolic integer-like automorphism with signature $\{p, q\}$. The characteristic polynomial $h(X)$ of f has integer coefficients and constant term ± 1. This implies that if $g(X)$ is a rational polynomial which divides $h(X)$, then it must have at least one root of absolute value strictly smaller than 1. We know that f induces an isomorphism on each quotient ${}^{\gamma_{i-1}(\mathfrak{n}^{\mathbb{Q}})}\!/\!{}_{\gamma_i(\mathfrak{n}^{\mathbb{Q}})}$ and thus the polynomial $h(X)$ has at least c irreducible factors. Therefore f has at least c eigenvalues of absolute value strictly smaller than 1 and thus $p \geq c$. By considering f^{-1} as well, we get that $q \geq c$ and this shows that $\min(\operatorname{sgn}(f)) \geq c$. We say that an Anosov automorphism $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ has minimal signature if equality holds, i.e. if $\min(\operatorname{sgn}(f)) = c$.

Existence of minimal signature

Question 9.1 asks if there exists Anosov automorphisms of minimal signature for $c = 2$ and already in Section 9.3.2 we gave a positive answer to this question as a consequence of the main theorem. So the existence of Anosov automorphisms of minimal signature is a generalization of Question 9.1 and with Theorem 9.14 we can also give a positive answer to the generalized question:

Theorem 9.22. *For every c, there exists an Anosov automorphism $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ on a Lie algebra of nilpotency class c such that f is of minimal signature.*

Proof. Let E be a real Galois extension of $\mathbb{Q}$ with $\operatorname{Gal}(E, \mathbb{Q}) \simeq \mathbb{Z}_{2n}$ for some $n > 1$ and σ a generator of $\operatorname{Gal}(E, \mathbb{Q})$. Take λ an unit Pisot number in E with the extra condition that $|\lambda \sigma^n(\lambda^2)| < 1$ as in Corollary 4.18. Since $n > 1$, we have that $|\lambda \sigma^n(\lambda)| > |\prod_{i=1}^{2n} \sigma^i(\lambda)| = 1$. Consider the collection of algebraic integers

$$\mu_{i,j} = \sigma^i(\lambda^{j-1})\sigma^{i+n}(\lambda)$$

for all $i \in \{1, \ldots, 2n\}$ and all $j \in \{1, \ldots, c\}$. The Galois conjugates $\sigma^i(\lambda)$ of λ are the $\mu_{i,j}$ with $j = 1$. Note that the definition implies that $\mu_{i+n,2} = \mu_{i,2}$ and all other $\mu_{i,j}$ are distinct because of the full rank condition.

Every $\mu_{i,j}$ with $i \notin \{n, 2n\}$ has absolute value < 1 since λ is a Pisot number. The algebraic unit $\mu_{n,3} = \lambda \sigma^n(\lambda^2)$ satisfies $|\mu_{n,3}| < 1$ because of our choice of λ and therefore also all $\mu_{n,j} = \sigma^n\left(\lambda^{j-3}\right)\mu_{n,3}$ with $j \geq 3$ have absolute value < 1. Thus it follows that of all $\mu_{i,j}$, only

$$\mu_{n,1} = \lambda, \mu_{n,2} = \mu_{2n,2} = \lambda\sigma^n(\lambda), \mu_{2n,3} = \lambda^2\sigma^n(\lambda), \ldots, \mu_{2n,c} = \lambda^{c-1}\sigma^n(\lambda)$$

have absolute value > 1, so in total there are c of the $\mu_{i,j}$ with $|\mu_{i,j}| > 1$.

Now consider the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ with basis $X_{\mu_{ij}}$ for all values of i and j, where we write the $\mu_{i,1}$ as the conjugates of λ. The Lie bracket on $\mathfrak{n}^{\mathbb{Q}}$ is given by on the one hand

$$[X_{\sigma^i(\lambda)}, X_{\sigma^{i+n}(\lambda)}] = X_{\mu_{i,2}}$$

for all $i \in \{1, \ldots, n\}$ and on the other hand by

$$[X_{\sigma^i(\lambda)}, X_{\mu_{i,j}}] = X_{\mu_{i,j+1}}$$

for all $i \in \{1, \ldots, 2n\}$, $j \in \{2, \ldots, c\}$ (and all other brackets are 0). It is easy to check that these relations define a Lie algebra (i.e. that the Jacobi identity holds) and that the 1-dimensional subspaces spanned by each basis vector satisfy the conditions of Corollary 9.15. The map ρ_σ given by $\rho_\sigma(X_{\mu_{i,j}}) = -X_{\sigma(\mu_{i,j})}$ for $i \in \{n, 2n\}$ and $j \geq 2$ and $\rho_\sigma(X_{\mu_{i,j}}) = X_{\sigma(\mu_{i,j})}$ for all other (i,j) defines a representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$. The minus sign in the first case comes from the relation $\mu_{i+n,2} = \mu_{i,2}$. The conditions of Corollary 9.15 are satisfied for ρ and thus this gives us a rational form $\mathfrak{m}^{\mathbb{Q}}$ with Anosov automorphism f. There are only c eigenvalues of f with absolute value > 1, so f is of minimal signature. □

Type of minimal signature

The type of the example constructed in this theorem is equal to

$$\underbrace{(2n, n, 2n, 2n, \ldots, 2n)}_{c \text{ components}}$$

for all $n \geq 2$. This is not the only possibility for Anosov automorphisms of minimal signature since one can construct examples on Lie algebras of type

$$\underbrace{(2n, n, 2n, n, 2n, \ldots)}_{c \text{ components}},$$

where the induced eigenvalues on $\gamma_{2j}(\mathfrak{n}^{\mathbb{Q}}) / \gamma_{2j+1}(\mathfrak{n}^{\mathbb{Q}})$ are of the form

$$\left(\sigma^i(\lambda)\, \sigma^{n+i}(\lambda)\right)^j.$$

The construction of such examples is similar as in Theorem 9.22.

We conjecture that these examples are the only possible types that can occur, see Chapter 11. The methods of this chapter are useful to prove or disprove this conjecture. In the remaining part of this section we prove this conjecture for some low nilpotency classes.

Note that if $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ is an Anosov automorphism of minimal signature on a Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ of nilpotency class c, then f also induces an Anosov automorphism of minimal signature on $\mathfrak{n}^{\mathbb{Q}} / \gamma_{i+1}(\mathfrak{n}^{\mathbb{Q}})$ of nilpotency class i for all $1 \leq i \leq c$. So it makes sense to first study the Anosov automorphisms of minimal signature for low nilpotency class. The abelian case, so this means the case where $\mathfrak{n}^{\mathbb{Q}} = \mathbb{Q}^n$ for some $n \geq 2$, is immediate: for every n, there exists an Anosov automorphism of minimal signature. So the first interesting case is nilpotency class 2.

It follows from Theorem 5.15 that if there exists an Anosov automorphism on a nilpotent Lie algebra $\mathfrak{n}^{\mathbb{Q}}$, there also exists an Anosov automorphism of the same signature which is diagonalizable over $\mathbb{C}$. Let $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ be diagonalizable and let f_i be the induced isomorphism on the quotient $\gamma_i(\mathfrak{n}^{\mathbb{Q}}) / \gamma_{i+1}(\mathfrak{n}^{\mathbb{Q}})$. If we denote the eigenvalues of f_1 by $\lambda_1, \ldots, \lambda_n$, then all eigenvalues of f_i can be written as an i-fold product $\lambda_{j_1} \ldots \lambda_{j_i}$.

Minimal signature in nilpotency class 2

By Theorem 9.22 there exists Anosov automorphisms of minimal signature on some Lie algebras of type $(2n, n)$ for all $n \geq 2$. We show that this is the only possibility for the type:

Proposition 9.23. *There exists a Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ of nilpotency class 2 and type (n_1, n_2) with an Anosov automorphism of minimal signature if and only if $n_1 = 2n_2$ and $n_2 \geq 2$.*

Proof. The existence was already shown above in the general situation. So it is sufficient to show that these conditions are really necessary. Of course the mere existence of an Anosov automorphism already implies that $n_2 \geq 2$, so we are left to prove that $n_1 = 2n_2$.

Let $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ be an Anosov automorphism of minimal signature and assume that f is diagonalizable over $\mathbb{C}$. By replacing f by its inverse if necessary, we can assume that f has exactly 2 eigenvalues with absolute value > 1. Let $p(X)$ be the characteristic polynomial of f and $q_1(X), q_2(X)$ the characteristic polynomial of the automorphism that f induces on $\mathfrak{n}^{\mathbb{Q}} / \gamma_2(\mathfrak{n}^{\mathbb{Q}})$ and $\gamma_2(\mathfrak{n}^{\mathbb{Q}})$ respectively, so $p(X) = q_1(X)q_2(X)$. Since $\min(\operatorname{sgn}(f)) = 2$, it follows that $q_1(X)$ and $q_2(X)$ are irreducible polynomials.

Let $\lambda_1, \ldots, \lambda_{n_1}$ be the roots of $q_1(X)$ with $\lambda_1 > 1$ and E the splitting field of $q_1(X)$. By the assumptions, there is a unique root of $q_2(X)$ with absolute value

> 1. All roots of $q_2(X)$ are of the form $\lambda_i\lambda_j$ and we claim that every λ_i occurs exactly once in such a product. If not, since all roots of $q_2(X)$ are distinct, this would imply that there exists three different integers i, j, k such that $\lambda_i\lambda_j$ and $\lambda_i\lambda_k$ are roots of $q_2(X)$. There exists an element $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ such that $\sigma(\lambda_i) = \lambda_1$, so after applying σ, we get that both $\lambda_1\lambda_j$ and $\lambda_1\lambda_k$ are roots of $q_2(X)$ for some $j \neq k$. But this is a contradiction, since both these roots would have absolute value > 1. So the claim must hold and from this observation it follows that $n_1 = 2n_2$. □

As explained before, the construction of Theorem 9.14 does not give us the isomorphism class of the constructed Anosov Lie algebra. It would be interesting to determine the isomorphism class of Lie algebras with Anosov automorphisms of minimal signature. The case of type $(4, 2)$ was already treated above.

Minimal signature in nilpotency class 3

Also in the case when $\mathfrak{n}^{\mathbb{Q}}$ has nilpotency class 3, we can completely determine the type of Lie algebras admitting an Anosov automorphism of minimal signature:

Proposition 9.24. *There exists a Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ of nilpotency class 3 and type (n_1, n_2, n_3) with an Anosov automorphism of minimal signature if and only if $n_1 = 2n_2 = n_3$ and $n_2 \geq 2$.*

Proof. The existence part was already done above. So we show that the conditions on the type are really necessary. From the proposition for nilpotency class 2, we already know that $n_1 = 2n_2$ and $n_2 \geq 2$.

Assume that $\mathfrak{n}^{\mathbb{Q}}$ is a nilpotent Lie algebra with $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ an Anosov automorphism of minimal signature and assume f is diagonalizable. Let $q_i(X)$ the characteristic polynomial of the automorphism that f induces on $\gamma_i(\mathfrak{n}^{\mathbb{Q}}) / \gamma_{i+1}(\mathfrak{n}^{\mathbb{Q}})$ and $p(X)$ the characteristic polynomial of f, so with

$$p(X) = q_1(X)q_2(X)q_3(X).$$

Since $\min(\mathrm{sgn}(f)) = 3$, it follows that every polynomial $q_i(X)$ is irreducible and has exactly one root with absolute value > 1.

Let $\lambda_1, \ldots, \lambda_{n_1}$ be the roots of $q_1(X)$ and assume that $\lambda_1 > 1$. As explained above, the roots of $q_3(X)$ are a 3-fold product of these eigenvalues λ_i. There are two possibilities, they can be of the form $\lambda_i\lambda_j\lambda_k$ with i, j, k distinct or of the form $\lambda_i^2\lambda_j$ with $i \neq j$. We show that the first possibility is impossible. Let X_i, X_j, X_k be the eigenvectors of $\mathfrak{n}^{\mathbb{C}}$ corresponding to the eigenvalues $\lambda_i, \lambda_j, \lambda_k$.

The eigenspace for the eigenvalue $\lambda_i\lambda_j\lambda_k$ is spanned by all vectors of the form $[X_{n_1},[X_{n_2},X_{n_3}]]$ with $\{n_1,n_2,n_3\} = \{i,j,k\}$ and thus these cannot be all 0. So after renumbering, we can assume that an eigenvector of $\lambda_i\lambda_j\lambda_k$ is given by $[X_i,[X_j,X_k]]$. From the Jacobi identity if then follows that $[X_i,X_k]$ and $[X_i,X_j]$ aren't both 0 and thus f would have two eigenvalues of absolute value > 1 on ${}^{\gamma_2(\mathfrak{n}^{\mathbb{Q}})}\!/\!_{\gamma_3(\mathfrak{n}^{\mathbb{Q}})}$ just like in the proof of the previous proposition.

So all the eigenvalues are of the form $\lambda_i^2\lambda_j$. Just as before it's easy to show that every λ_i occurs exactly once in a product $\lambda_i^2\lambda_j$ and once in a product $\lambda_i\lambda_k^2$ since f is of minimal signature. So this implies $n_3 = n_1$ and concludes the proof. □

Minimal signature in higher nilpotency class

The higher the nilpotency class, the harder it gets to determine the type with simple number theoretical arguments like in the two propositions above. Also, the number of possibilities further increases. We demonstrate this by stating the generalization of the previous propositions for $c = 4$ without proof.

For nilpotency class 4, the characteristic polynomial of an Anosov of minimal signature is of the form

$$p(X) = q_1(X)q_2(X)q_3(X)q_4(X)$$

with the definition of the q_i as above. For the roots of $q_4(X)$ there will be two possibilities: they can all be of the form $\lambda_i^2\lambda_j^2$ or of the form $\lambda_i^3\lambda_j$ with $i \neq j$. The first possibility gives us that $n_4 = n_2$, the second one that $n_4 = n_1$. The construction of Theorem 9.22 corresponds to the second possibility.

Proposition 9.25. *There exists a nilpotent Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ of nilpotency class 4 and type $(n_1,\ldots,n_4)$ with an Anosov automorphism f of minimal signature if and only if there exists a natural number $n \geq 2$ such that the type is of the form*

(1) $(n_1,\ldots,n_4) = (2n,n,2n,n)$ *or*

(2) $(n_1,\ldots,n_4) = (2n,n,2n,2n)$.

The proof is left as an exercise for the reader and is similar to the proofs for nilpotency class 2 and 3. The existence of the first case follows again easily from the main theorem.

For higher nilpotency classes, the situation gets even more diverse, although the existence question still follows from the main theorem. Starting from a Pisot number λ and the form of the eigenvalues on ${}^{\gamma_i(\mathfrak{n}^{\mathbb{Q}})}\!/\!{}_{\gamma_{i+1}(\mathfrak{n}^{\mathbb{Q}})}$ (as a product of Galois conjugates of λ), Theorem 9.14 guarantees us the existence of an Anosov automorphism of minimal signature.

9.3.4 Anosov Lie algebras of minimal type

In this section we first recall a lower bound for the type of an Anosov Lie algebra. With our general construction we can give examples to show that this lower bound is always attained.

Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational Lie algebra of type $(n_1, \ldots, n_c)$ which is Anosov and take $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ a hyperbolic and integer-like automorphism on $\mathfrak{n}^{\mathbb{Q}}$ with characteristic polynomial $p(X)$. There are no rational polynomials of degree 1 dividing $p(X)$ and so every non-constant rational polynomial dividing $p(X)$ has at least degree 2. Since the map f also induces an isomorphism f_i on ${}^{\gamma_{i-1}(\mathfrak{n}^{\mathbb{Q}})}\!/\!{}_{\gamma_i(\mathfrak{n}^{\mathbb{Q}})}$ and the characteristic polynomial of f_i divides $p(X)$, it must hold that $n_i \geq 2$ for all $i \in \{1, \ldots, c\}$.

Note that $n_2 \leq \frac{n_1(n_1-1)}{2}$ holds for every Lie algebra and thus if $n_1 = 2$ the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ must be abelian. If $n_1 = 4$ then the lower bound $n_i \geq 2$ for $i \in \{2, \ldots, c\}$ is optimal, as was shown in [72] as a consequence of Theorem 9.17. In fact the examples of [72] also form Anosov Lie algebras of minimal dimension for every nilpotency class c. The question remains how a lower bound for $n_1 = 3$ looks like.

So assume that $n_1 = 3$ for the Anosov Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ and denote by $\lambda_1, \lambda_2, \lambda_3$ the eigenvalues of the induced map on ${}^{\mathfrak{n}^{\mathbb{Q}}}\!/\!{}_{\gamma_2(\mathfrak{n}^{\mathbb{Q}})}$. Take $E = \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)$, then E is Galois over $\mathbb{Q}$ of degree 3 or 6. If μ is an eigenvalue of the induced map on ${}^{\gamma_i(\mathfrak{n}^{\mathbb{Q}})}\!/\!{}_{\gamma_{i+1}(\mathfrak{n}^{\mathbb{Q}})}$, then μ is an i-fold product of the eigenvalues λ_j and thus $\mathbb{Q} \subseteq \mathbb{Q}(\mu) \subseteq E$. So there are 4 possibilities for the degree $[\mathbb{Q}(\mu) : \mathbb{Q}]$, namely $1, 2, 3$ or 6. We claim that these first two possibilities are impossible.

The first case is impossible, since this would imply that $\mu = \pm 1$ and therefore f would have an eigenvalue of absolute value 1. For the second case, assume that $[\mathbb{Q}(\mu) : \mathbb{Q}] = 2$ and thus that $\mathrm{Gal}(E, \mathbb{Q}) = S_3$. Write $\mu = \lambda_1^k \lambda_2^l \lambda_3^m$ with $k, l, m \in \mathbb{N}$. By using the relation $\lambda_1 \lambda_2 \lambda_3 = 1$, we can assume that only two eigenvalues λ_i occur in μ, so after renumbering we get $\mu = \lambda_1^k \lambda_2^l$. Take $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ the unique element with $\sigma(\lambda_1) = \lambda_2$ and $\sigma(\lambda_2) = \lambda_3$. Since $[\mathbb{Q}(\mu) : \mathbb{Q}] = 2$ and σ generates the unique subgroup of $\mathrm{Gal}(E, \mathbb{Q})$ of order 3, the Galois correspondence implies that $\sigma(\mu) = \mu$. So

$$\mu = \lambda_1^k \lambda_2^l = \sigma(\mu) = \lambda_2^k \lambda_3^l = \sigma^2(\mu) = \lambda_3^k \lambda_1^l$$

but this is a contradiction since at least one of these products has absolute value > 1 and at least one has absolute value < 1.

So we conclude that 3 divides $[\mathbb{Q}(\mu) : \mathbb{Q}]$ and thus 3 also divides the degree of the minimal polynomial of μ over $\mathbb{Q}$. This implies that $3 \mid n_i$ and in particular, it must hold that $n_i \geq 3$. A similar reasoning can be used to show that e.g. type $(5, 3)$ can only occur with an abelian factor, simplifying the proof given in [74].

In summary, we have the following possibilities for the type of an Anosov Lie algebra.

Theorem 9.26. *Let $\mathfrak{n}^{\mathbb{Q}}$ be an Anosov Lie algebra of type $(n_1, \ldots, n_c)$, then it must hold that :*

(1) *$\mathfrak{n}^{\mathbb{Q}}$ is abelian or*

(2) *$3 \mid n_i$ for all $i = 1, \ldots, c$ or*

(3) *$n_1 \geq 4$ and $n_i \geq 2$ for all $i = 2, \ldots, c$.*

The original proofs of this result can be found in the papers [82] and [89].

Question 9.2 asks if the second possibility is possible for every nilpotency class c. We state this theorem in a more general setting:

Theorem 9.27. *For every $c \in \mathbb{N}_0$ and $n \in \mathbb{N}$ with $n > 2$, there exists an Anosov Lie algebra of type $(n, \ldots, n)$ and nilpotency class c.*

Proof. Let $E \supseteq \mathbb{Q}$ be a real Galois extension with cyclic Galois group of order n and let $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ be a generator. Take $\lambda_1 \in E$ an unit Pisot number and consider the Galois conjugates $\lambda_1, \lambda_2 = \sigma(\lambda_1), \ldots, \lambda_n = \sigma^{n-1}(\lambda_1)$. Define the algebraic units

$$\mu_{i,j} = \lambda_i^{j-1} \sigma(\lambda_i)$$

for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, c\}$, where the algebraic units λ_i occur as the $\mu_{i,j}$ with $j = 1$. Let $\mathfrak{n}^{\mathbb{Q}}$ be the Lie algebra with basis $X_{\mu_{i,j}}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, c\}$, with Lie bracket given by

$$[X_{\lambda_i}, X_{\mu_{i,j}}] = X_{\mu_{i,j+1}}$$

for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, c\}$ and all other brackets 0. It is easy to see that the Jacobi identity holds and thus that $\mathfrak{n}^{\mathbb{Q}}$ is indeed a Lie algebra.

The linear map $h : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ defined by $h(X_{\mu_{i,j}}) = X_{\sigma(\mu_{i,j})}$ is an automorphism of this Lie algebra of order n. So the map $\rho : \sigma \mapsto h$ defines a representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$. This Lie algebra and the representation ρ satisfy the conditions of Corollary 9.15 (where the spaces V_λ are one-dimensional and spanned by the basis vectors) and thus there exists a rational form of $\mathfrak{n}^E$ which is Anosov. The type of this rational form is equal to the type of $\mathfrak{n}^{\mathbb{Q}}$. $\square$

The case where $n = 3$ gives an answer to Question 9.2. By using Corollary 9.15, it is easy to construct examples of other types than $(3, 3, \ldots, 3)$.

Example 9.28. Let E be a Galois extension of $\mathbb{Q}$ with $\mathrm{Gal}(E, \mathbb{Q}) \simeq \mathbb{Z}_3$. Denote with $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ a generator of the Galois group. Take λ a hyperbolic unit in E and write $\lambda_1 = \lambda, \lambda_2 = \sigma(\lambda)$ and $\lambda_3 = \sigma^2(\lambda)$. Consider the rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ with basis $X_{\lambda_i}, Y_{\lambda_i \lambda_j}$ and $Z_{\lambda_i^2 \lambda_j}$ for $i, j \in \{1, 2, 3\}$ with i and j distinct. Since λ satisfies the full rank condition, all the algebraic integers used as an index for the basis vectors are distinct. The relations between the basis vectors are given by

$$[X_{\lambda_i}, X_{\lambda_j}] = Y_{\lambda_i \lambda_j}$$

$$[X_{\lambda_i}, Y_{\lambda_i \lambda_j}] = Z_{\lambda_i^2 \lambda_j},$$

for $i \neq j$. Note that $\mathfrak{n}^{\mathbb{Q}}$ is the free nilpotent Lie algebra of nilpotency class 3 on three generators X_1, X_2, X_3 divided out by the 2-dimensional ideal spanned by the vectors $[X_i, [X_j, X_k]]$ for distinct $i, j, k \in \{1, 2, 3\}$. The Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ has type $(3, 3, 6)$ and $\mathfrak{n}^E$ has a rational form which is Anosov by Corollary 9.15, where the representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ is given by

$$\rho_\sigma \left(X_{\lambda_i} \right) = X_{\sigma(\lambda_i)}$$

$$\rho_\sigma \left(Y_{\lambda_i \lambda_j} \right) = Y_{\sigma(\lambda_i \lambda_j)}$$

$$\rho_\sigma \left(Z_{\lambda_i^2 \lambda_j} \right) = Z_{\sigma(\lambda_i^2 \lambda_j)}.$$

It is an open question to determine all possibilities for the types $(n_1, \ldots, n_c)$ of Anosov Lie algebras with $n_1 = n_2 = 3$. To solve this problem, a careful study of the conjugates of algebraic units of degree 3 is needed, see Chapter 11. When $c = 3$, there are only 2 possibilities, namely type $(3, 3, 3)$ and type $(3, 3, 6)$.

9.3.5 Anosov Lie algebra without expanding automorphism

As another application of Theorem 8.17 and Theorem 9.14, we construct a nilmanifold which admits an Anosov diffeomorphism but no non-trivial self-cover and so also no expanding map. This is the first example of a nilmanifold satisfying these properties. On the other hand, it is easy to give examples of nilmanifolds admitting an expanding map but no Anosov diffeomorphism, e.g. the nilmanifold $H_3(\mathbb{Z})\backslash H_3(\mathbb{R})$.

First, we will construct a rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ as a quotient of a free nilpotent Lie algebra $\mathfrak{g}^{\mathbb{Q}}$ by using the Hall basis of such a Lie algebra. Next, we give a general way of constructing automorphisms on this Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. In this way, we give a finite group of automorphisms H such that there exist no partially expanding automorphism of $\mathfrak{n}^{\mathbb{Q}}$ commuting with H. By Theorem 8.13 and Theorem 8.17 this shows that there are no partially expanding automorphisms on this Lie algebra nor on any rational form of the Lie algebra $\mathfrak{n}^{\mathbb{R}}$. Finally, we use the techniques of [41] to prove that the Lie algebra $\mathfrak{n}^{\mathbb{R}}$ has a rational form which is Anosov. So every nilmanifold corresponding to this rational form then has the desired properties.

Hall basis

Let $\mathfrak{g}^{\mathbb{Q}}$ be the free 6-step nilpotent Lie algebra over $\mathbb{Q}$ on 4 generators. Denote by X_1, X_2, X_3, X_4 a set of generators for the Lie algebra $\mathfrak{g}^{\mathbb{Q}}$. Consider the natural grading $\mathfrak{g}_1^{\mathbb{Q}} \oplus \ldots \oplus \mathfrak{g}_6^{\mathbb{Q}}$ for $\mathfrak{g}^{\mathbb{Q}}$, where $\mathfrak{g}_1^{\mathbb{Q}}$ is the vector space spanned by $X_1, \ldots, X_4$. We say that the vector a has degree i if $a \in \mathfrak{g}_i^{\mathbb{Q}}$ and we denote this as $\deg(a) = i$. We will use the shorthand notation $[a, b, c]$ with $a, b, c \in \mathfrak{g}^{\mathbb{Q}}$ for the Lie bracket $[a, [b, c]]$ and similarly for longer brackets. If Y and Z are subsets of $\mathfrak{g}^{\mathbb{Q}}$, then we write $[Y, Z]$ for the Lie algebra generated by all elements $[y, z]$ with $y \in Y, z \in Z$. The Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ we construct will be a double quotient of this free Lie algebra $\mathfrak{g}^{\mathbb{Q}}$.

A explicit basis for the Lie algebra $\mathfrak{g}^{\mathbb{Q}}$ as vector space over $\mathbb{Q}$ is the Hall basis. The elements of this basis are constructed inductively: given the basis for $\mathfrak{g}_i^{\mathbb{Q}}$ with $1 \leq i \leq k$, we build the basis for $\mathfrak{g}_{k+1}^{\mathbb{Q}}$. We fix an order relation on the basis vectors of $\mathfrak{g}_1^{\mathbb{Q}}, \ldots, \mathfrak{g}_k^{\mathbb{Q}}$, assuming that $a < b$ if $\deg(a) < \deg(b)$. The basis vectors for $\mathfrak{g}_{k+1}^{\mathbb{Q}}$ are then given by Lie brackets $[a, b] \in \mathfrak{g}_{k+1}^{\mathbb{Q}}$ with $\deg(a) + \deg(b) = k + 1$ and $a < b$ with the extra condition that if $b = [b_1, b_2]$ then $a \geq b_1$. For more details and a proof that these vectors indeed form a basis, we refer to [58]. In our case we will always assume that the order relation satisfies $X_1 < X_2 < X_3 < X_4$

on $\mathfrak{g}_1^{\mathbb{Q}}$, so $X_{i_1} < X_{i_2}$ if and only if $i_1 < i_2$. From now on we will fix a Hall basis for $\mathfrak{g}^{\mathbb{Q}}$ and an order relation on the basis vectors.

Denote by

$$\mathfrak{g}' = [\mathfrak{g}^{\mathbb{Q}}, \mathfrak{g}^{\mathbb{Q}}] = \mathfrak{g}_2^{\mathbb{Q}} \oplus \ldots \oplus \mathfrak{g}_6^{\mathbb{Q}}$$

the derived subalgebra of $\mathfrak{g}^{\mathbb{Q}}$ and take the subalgebra $\mathfrak{m}^{\mathbb{Q}} = [\mathfrak{g}', \mathfrak{g}'] \subseteq \mathfrak{g}_4^{\mathbb{Q}} \oplus \mathfrak{g}_5^{\mathbb{Q}} \oplus \mathfrak{g}_6^{\mathbb{Q}}$ which is an ideal of $\mathfrak{g}^{\mathbb{Q}}$. It is well known that the elements $[X_{i_1}, \ldots, X_{i_k}]$ with $i_1 \geq \ldots \geq i_{k-1} < i_k$ of the Hall basis for $\mathfrak{g}^{\mathbb{Q}}$ project to a basis for the quotient $\mathfrak{g}^{\mathbb{Q}} / \mathfrak{m}^{\mathbb{Q}}$. Therefore all the other elements of the Hall basis form a basis for $\mathfrak{m}^{\mathbb{Q}}$ as a vector space.

Lemma 9.29. *Let β, γ and δ be the Hall bases of $\mathfrak{g}_2^{\mathbb{Q}}$, $\mathfrak{g}_3^{\mathbb{Q}}$ and $\mathfrak{g}_4^{\mathbb{Q}}$ respectively. Then the Hall basis of $\mathfrak{g}_6^{\mathbb{Q}} \cap \mathfrak{m}^{\mathbb{Q}}$ is given by the set $B = B_1 \cup B_2$ where*

$$B_1 = \big\{ [b,d] \mid b \in \beta,\ d \in \delta,\ d \neq [b_1, b_2] \textit{ with } b_i \in \beta, b_1 > b \big\}$$

$$B_2 = \big\{ [c_1, c_2] \mid c_i \in \gamma,\ c_1 < c_2 \big\}.$$

Note that if $b < b_1 < b_2$, then the Jacobi identity gives us

$$[b, b_1, b_2] = [b_1, b, b_2] - [b_2, b, b_1]$$

and the last two vectors are elements of the Hall basis. This implies that every element of $[\mathfrak{g}_2^{\mathbb{Q}}, \mathfrak{g}_4^{\mathbb{Q}}]$ can be written as a linear combination of elements in B_1. Similarly every element of $[\mathfrak{g}_3^{\mathbb{Q}}, \mathfrak{g}_3^{\mathbb{Q}}]$ can be written as a linear combination of elements in B_2.

Proof. The only other possibility for basis vectors in $\mathfrak{g}_6^{\mathbb{Q}} \cap \mathfrak{m}^{\mathbb{Q}}$ is $[X_i, e]$ with e in the Hall basis for $\mathfrak{g}_5^{\mathbb{Q}}$. Note that e is not given by the Lie bracket between two vectors of degree 2 and 3, since $[X_i, e]$ is in the Hall basis. So e is of the form $[X_j, d]$ with $d \in \delta$. Again, the vector d is not equal to the Lie bracket of two vectors of degree 2 since $[X_j, d]$ is in the Hall basis. So d is of the form $d = [X_k, c]$ with $c \in \gamma$. But this implies that $[X_i, e] = [X_i, X_j, X_k, c] \notin \mathfrak{m}^{\mathbb{Q}}$. $\square$

Construction of the Lie algebra $\mathfrak{n}$

Let $\mathfrak{g}^{\mathbb{Q}}$ be the free nilpotent Lie algebra on 4 generators over $\mathbb{Q}$ as in the previous paragraph. Every permutation $s \in S_4$ determines an automorphism φ_s of $\mathfrak{g}^{\mathbb{Q}}$ which is given by the relations $\varphi_s(X_i) = X_{s(i)}$ on the generators. Consider the automorphism $\alpha \in \operatorname{Aut}(\mathfrak{g}^{\mathbb{Q}})$ of order 4 which is induced by the permutation $(1234) \in S_4$.

Let I be the smallest ideal of $\mathfrak{g}^{\mathbb{Q}}$ such that

$$[X_i, X_1, X_3], [X_i, X_2, X_4] \text{ and } [X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}] \in I$$

for all $i, i_j \in \{1, \ldots, 4\}$ with the indices i_j distinct. From the Jacobi identity it follows that also

$$[[X_{i_1}, X_{i_2}], X_{i_3}, X_{i_4}] = -[X_{i_3}, X_{i_4}, X_{i_1}, X_{i_2}] + [X_{i_4}, X_{i_3}, X_{i_1}, X_{i_2}] \in I$$

for all distinct $i_j \in \{1, \ldots, 4\}$. The ideal I is a graded ideal, meaning that I is the direct sum of the vector spaces $I_k = I \cap \mathfrak{g}_k^{\mathbb{Q}}$. Since the image of a generator of I under α is up to sign again a generator of I, it follows that I is invariant under α, i.e. $\alpha(I) = I$. Let $\tilde{\mathfrak{n}}^{\mathbb{Q}}$ be the quotient Lie algebra ${}^{\mathfrak{g}^{\mathbb{Q}}}\!/_{I}$ and denote by $\tilde{p} : \mathfrak{g}^{\mathbb{Q}} \to \tilde{\mathfrak{n}}^{\mathbb{Q}}$ the natural projection map. Since I is invariant under α, the automorphism α induces an automorphism of the quotient $\tilde{\mathfrak{n}}^{\mathbb{Q}}$. Denote this induced map as $\tilde{\alpha} : \tilde{\mathfrak{n}}^{\mathbb{Q}} \to \tilde{\mathfrak{n}}^{\mathbb{Q}}$.

Consider the vector $v = [[X_4, X_3, X_4], X_2, X_1, X_2] \in \mathfrak{g}_6^{\mathbb{Q}}$ and write $\tilde{p}(v) = w$. The vector space spanned by $\tilde{p}\,(v - [X_2, X_4]) = w - \tilde{p}([X_2, X_4]) \in \tilde{\mathfrak{n}}^{\mathbb{Q}}$ is an ideal by definition of the Lie algebra $\tilde{\mathfrak{n}}^{\mathbb{Q}}$. Let J be the smallest ideal of $\tilde{\mathfrak{n}}^{\mathbb{Q}}$ containing this vector and which is invariant under $\tilde{\alpha}$. Since $\tilde{\alpha}^2(v) = -v$ the dimension of J is equal to 2. Consider the Lie algebra $\mathfrak{n} = {}^{\tilde{\mathfrak{n}}^{\mathbb{Q}}}\!/_{J}$ with projection map $p : \mathfrak{g}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$. The automorphism $\tilde{\alpha}$ induces an automorphism $\bar{\alpha} \in \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ of order 4.

We start by showing that $\tilde{p}(v) = w \neq 0$.

Lemma 9.30. *The vector v satisfies $\tilde{p}(v) \neq 0$ for $\tilde{p} : \mathfrak{g}^{\mathbb{Q}} \to \tilde{\mathfrak{n}}^{\mathbb{Q}}$ the projection map as above.*

Proof. Since $v \in \mathfrak{m}^{\mathbb{Q}} \cap \mathfrak{g}_6^{\mathbb{Q}}$, it suffices to show that $v \notin I_6 \cap \mathfrak{m}^{\mathbb{Q}}$. We express the generators of $I_6 \cap \mathfrak{m}^{\mathbb{Q}}$ in terms of the Hall basis $B = B_1 \cup B_2$ of $\mathfrak{g}_6^{\mathbb{Q}} \cap \mathfrak{m}^{\mathbb{Q}}$ as explained in Lemma 9.29. Note that v or $-v$ is an element of B_2.

From the Jacobi identity, it follows that the vector space $I_6 \cap \mathfrak{m}^{\mathbb{Q}}$ satisfies

$$I_6 \cap \mathfrak{m}^{\mathbb{Q}} = [\mathfrak{g}_2^{\mathbb{Q}}, I_4] + [\mathfrak{g}_3^{\mathbb{Q}}, I_3] + [\mathfrak{g}_1^{\mathbb{Q}}, \mathfrak{g}_1^{\mathbb{Q}}, I_4 \cap \mathfrak{m}^{\mathbb{Q}}].$$

The elements of $[\mathfrak{g}_2^{\mathbb{Q}}, I_4]$ are linear combinations of elements in B_1. Let γ be the Hall basis for $\mathfrak{g}_3^{\mathbb{Q}}$, then the vector space $[\mathfrak{g}_3^{\mathbb{Q}}, I_3]$ is spanned by vectors of the form $[c, [X_i, X_1, X_3]]$ and $[c, [X_i, X_2, X_4]]$ for $c \in \gamma$ and $1 \leq i \leq 4$. These vectors can easily be expressed in terms of the elements of B_2. This already implies that $v \notin [\mathfrak{g}_3^{\mathbb{Q}}, I_3]$.

To describe the generators of $[\mathfrak{g}_1^{\mathbb{Q}}, \mathfrak{g}_1^{\mathbb{Q}}, I_4 \cap \mathfrak{m}^{\mathbb{Q}}]$ in the Hall basis, note that

$$[X_j, b_1, b_2] = [b_1, X_j, b_2] - [b_2, X_j, b_1]$$

and thus

$$[X_i, X_j, b_1, b_2] = [b_1, X_i, X_j, b_2] + [[X_i, b_1], [X_j, b_2]]$$
$$- [[X_i, b_2], [X_j, b_1]] - [b_2, X_i, X_j, b_1]$$

for all $[b_1, b_2] \in I_4$. By expressing the vectors $[b_1, X_i, X_j, b_2]$ and $[b_2, X_i, X_j, b_1]$ in the Hall basis, we only get elements of B_1 by using the remark under Lemma 9.29. On the other hand, the vector $[[X_i, b_1], [X_j, b_2]] - [[X_i, b_2], [X_j, b_1]]$ is expressed only in terms of vectors in X_2. The only way of getting the vector $\pm v$ in this expression is in the case where $i = 4, j = 2, b_1 = \pm[X_3, X_4]$ and $b_2 = \pm[X_1, X_2]$ or in the situation with i, j and b_1, b_2 interchanged. In both cases, the other vector is up to sign equal to

$$[[X_2, X_3, X_4], X_4, X_1, X_2] = [[X_3, X_2, X_4], X_4, X_1, X_2]$$
$$- [[X_4, X_2, X_3], X_4, X_1, X_2],$$

where these last two vectors are up to sign in X_2. The first vector lies in $[\mathfrak{g}_3^{\mathbb{Q}}, I_3]$, but the second vector $[[X_4, X_2, X_3], X_4, X_1, X_2]$ is not an element of $[\mathfrak{g}_3^{\mathbb{Q}}, I_3]$. This last statement can be checked by using the expression of the generators of $[\mathfrak{g}_3^{\mathbb{Q}}, I_3]$ in terms of the basis B_2. We conclude that $v \notin I$. □

From Lemma 9.30 it follows that $p(v) \neq 0$.

Lemma 9.31. *Let $v = [[X_4, X_3, X_4], X_2, X_1, X_2] \in \mathfrak{g}^{\mathbb{Q}}$ and $\mathfrak{n}^{\mathbb{Q}}$ the Lie algebra as above with projection map $p : \mathfrak{g}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$. Then $p(v) \neq 0$.*

Proof. The vector $\tilde{p}(v) \neq 0$ by Lemma 9.30. Therefore the vectors $\tilde{p}(v), \tilde{p}([X_1, X_3])$ and $\tilde{p}([X_2, X_4])$ are linearly independent in $\tilde{\mathfrak{n}}^{\mathbb{Q}}$ since I is a graded ideal and $I_2 = I \cap \mathfrak{g}_2^{\mathbb{Q}} = 0$. Assume that $p(v) = 0$, then the ideal J contains the vectors $\tilde{p}(v), \tilde{p}([X_1, X_3])$ and $\tilde{p}([X_2, X_4])$. This is impossible since the dimension of J is equal to 2. □

Automorphisms on $\mathfrak{n}^{\mathbb{Q}}$

By the explicit construction of the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ as a quotient of the free Lie algebra $\mathfrak{g}^{\mathbb{Q}}$ we can give a general way of constructing automorphisms on $\mathfrak{n}^E$ for any field extension $E \supseteq \mathbb{Q}$. Consider the linear subspace $\mathfrak{g}_1^E$ of $\mathfrak{g}^E$ spanned by $X_1, \ldots, X_4$. Take $\lambda_1, \ldots, \lambda_4 \in E$ such that $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$ and consider the linear map $\mathfrak{g}_1^E \to \mathfrak{g}_1^E$ given by

$$X_i \mapsto \lambda_i X_i.$$

This map uniquely extends to an automorphism $\varphi : \mathfrak{g}^E \to \mathfrak{g}^E$. Moreover, since I is generated by eigenvectors of φ, it also induces an automorphism $\tilde{\varphi} : \tilde{\mathfrak{n}}^E \to \tilde{\mathfrak{n}}^E$. The vector $\tilde{p}(v - [X_2, X_4]) \in \tilde{\mathfrak{n}}^E$ is also an eigenvector of the map φ since $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$ and thus $\tilde{\varphi}$ induces a map $\bar{\varphi}$ on the Lie algebra $\mathfrak{n}^E$.

Take the basis $\bar{X}_1, \ldots, \bar{X}_4$ for the vector space ${}^{\mathfrak{n}^E}\!\big/_{[\mathfrak{n}^E, \mathfrak{n}^E]}$. Under the natural projection map $\pi : \operatorname{Aut}(\mathfrak{n}^E) \to \operatorname{Aut}\left({}^{\mathfrak{n}^E}\!\big/_{[\mathfrak{n}^E, \mathfrak{n}^E]}\right) \simeq \mathrm{GL}(4, E)$, the automorphism $\bar{\varphi}$ is mapped to the diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_4. This will be an important construction of automorphisms on the Lie algebra $\mathfrak{n}^E$.

As a consequence of this construction for automorphisms, we have the following proposition:

Proposition 9.32. *The Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ has no partially expanding automorphisms.*

Proof. Let H be the subgroup of $\mathrm{GL}(4, \mathbb{Q})$ generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This subgroup is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the centralizer of H in $\mathrm{GL}(4, \mathbb{Q})$ is given by the diagonal matrices $D(n, \mathbb{Q})$. As described just above this theorem, each of the generators of H above induces an automorphism of the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ and thus we get a faithful representation $i : H \to \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}})$.

Assume that $\mathfrak{n}^{\mathbb{Q}}$ does have a partially expanding automorphism φ. By Theorem 8.17, we can assume that φ commutes with every element of the finite group $i(H)$. Consider the vector space ${}^{\mathfrak{n}^{\mathbb{Q}}}\!\big/_{[\mathfrak{n}^{\mathbb{Q}}, \mathfrak{n}^{\mathbb{Q}}]}$ with basis $\bar{X}_1, \ldots, \bar{X}_4$ and the natural projection map $\pi : \operatorname{Aut}(\mathfrak{n}^{\mathbb{Q}}) \to \operatorname{Aut}\left({}^{\mathfrak{n}^{\mathbb{Q}}}\!\big/_{[\mathfrak{n}^{\mathbb{Q}}, \mathfrak{n}^{\mathbb{Q}}]}\right) \simeq \mathrm{GL}(4, \mathbb{Q})$. Since $\pi(\varphi)$ lies in the centralizer of H, we know that

$$\pi(\varphi) = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

The vector $p(v)$, with v as in the definition of $\mathfrak{n}^{\mathbb{Q}}$ above, is then an eigenvector of φ with eigenvalue $\lambda_1\lambda_2^2\lambda_3\lambda_4^2$ and $p([X_2, X_4])$ is an eigenvector of eigenvalue $\lambda_2\lambda_4$. Since $p(v) = p([X_2, X_4]) \neq 0$, it must hold that $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$. This is a contradiction since φ is a partially expanding automorphism. $\square$

Anosov Lie algebra

As an application of Theorem 9.14, we have the following consequence:

Proposition 9.33. *The Lie algebra $\mathfrak{n}^{\mathbb{R}}$ has a rational form which is Anosov.*

Proof. Take $\mathbb{Q} \subseteq E \subseteq \mathbb{R}$ a field extension with Galois group $\mathrm{Gal}(E, \mathbb{Q}) \simeq \mathbb{Z}_4$ and denote by σ a generator of $\mathrm{Gal}(E, \mathbb{Q})$. Let μ be a unit Pisot number in E and write $\mu_i = \sigma^{i-1}(\mu)$. By squaring μ if necessary, we can assume that $\mu_1\mu_2\mu_3\mu_4 = 1$. Take $\varphi : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ the automorphism induced by the linear map that maps

$$X_i \to \mu_i X_i,$$

as explained above Proposition 9.32.

All eigenvalues of the automorphism φ are products of the algebraic units μ_i of length at most 6. The only possibility to get an eigenvalue of absolute value 1 is $\mu_1\mu_2\mu_3\mu_4$, since μ satisfies the full rank condition, see above. By construction of the Lie algebra $\tilde{\mathfrak{n}}^{\mathbb{Q}}$, all the eigenvectors with eigenvalue $\mu_1\mu_2\mu_3\mu_4$ lie in I, so this eigenvalues does not occur. Hence, φ has no eigenvalues of absolute value 1.

Consider the representation $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}^{\mathbb{Q}})$ given by $\rho(\sigma) = \bar{\alpha}$. The maps ρ and φ satisfy the conditions of Corollary 9.15, where the subspaces V_λ are given the eigenspaces of the map φ and the map f of the theorem is equal to φ. This implies that $\mathfrak{n}^E$ (and therefore also $\mathfrak{n}^{\mathbb{R}}$) has a rational form which is Anosov. $\square$

Chapter 10

Infra-nilmanifolds modeled on a free nilpotent Lie group

The main theorem of this chapter gives an algebraic way of deciding whether an infra-nilmanifolds modeled on a free nilpotent Lie group admits an Anosov diffeomorphism or not. The exact statement of the algebraic description is as follows, see Main Theorem 6.

Theorem 10.1. *Let $\Gamma\backslash G$ be an infra-nilmanifold modeled on a free c-step nilpotent Lie group G and consider the associated abelianized rational holonomy representation $\overline{\varphi} : F \to \operatorname{Aut}\left({}^{N_{\mathbb{Q}}}\!\big/\!{}_{[N_{\mathbb{Q}}, N_{\mathbb{Q}}]} \right)$ where F is the holonomy group of Γ. Then the following statements are equivalent:*

$$\Gamma\backslash G \textit{ admits an Anosov diffeomorphism.}$$
$$\Updownarrow$$
$$\textit{Every } \mathbb{Q}\textit{-irreducible component of } \bar{\varphi} \textit{ that occurs with multiplicity } m,$$
$$\textit{splits in more than } \tfrac{c}{m} \textit{ components when seen as a representation over } \mathbb{R}.$$

We will explain what the abelianized rational holonomy representation is in Section 10.1. This result shows that the existence of an Anosov diffeomorphism only depends on this abelianized rational holonomy representation and more specifically on how this representation splits over the rationals $\mathbb{Q}$ and the reals $\mathbb{R}$. If the character table of the holonomy group F is known, this result gives us an effective way of checking whether the infra-nilmanifold admits an Anosov diffeomorphism or not.

The first version of this theorem was given by H.L. Porteous in [90] for flat manifolds, which are exactly the infra-nilmanifolds modeled on the free abelian Lie groups $\mathbb{R}^n$. The case where the holonomy group F is abelian was given by K. Dekimpe and K. Verheyen in [40, Theorem 2.3.].

The proof of Theorem 10.1 combines different techniques of representation theory for finite groups with some results in number theory, which were already proved in Chapter 4. The first section explains the methods of [40] for abelian groups and how this idea can be generalized to arbitrary groups F. The second section discusses the necessary results concerning representations of finite groups, in particular about minimal splitting fields for representations. After the proof in the third section, there are some applications of Theorem 10.1 in Section 10.4. In Section 10.5 we show that this result can be generalized to an even bigger class of infra-nilmanifolds.

10.1 Reduction to representation theory of finite groups

We start by recalling what a free nilpotent Lie group is. We say that a real Lie algebra $\mathfrak{g}$ is a free nilpotent Lie algebra of nilpotency class c on n generators if there exist $X_1, \ldots, X_n \in \mathfrak{g}$ such that for every map $\{X_1, \ldots, X_n\} \to \mathfrak{h}$ with $\mathfrak{h}$ a nilpotent Lie algebra of nilpotency class $\leq c$, there exists a unique extension $\mathfrak{g} \to \mathfrak{h}$ which is a Lie algebra morphism. For every n and c, there exists a c-step nilpotent Lie algebra on n generators and this Lie algebra is unique up to isomorphism. We denote this Lie algebra as $\mathfrak{g}_{n,c}$. A simply connected and connected Lie group G is called a free nilpotent Lie group of nilpotency class c on n generators if the Lie algebra corresponding to G is isomorphic to $\mathfrak{g}_{n,c}$ and the Lie group G is in that case written as $G_{n,c}$.

Let $X_1, \ldots, X_n$ be n generators for the free nilpotent Lie algebra $\mathfrak{g}_{n,c}$ and consider V the vector space spanned by $X_1, \ldots, X_n$. Every linear map $\varphi \in \operatorname{Aut}(V)$ uniquely extends to a Lie algebra automorphism by the definition of $\mathfrak{g}_{n,c}$. The reason for studying Anosov diffeomorphisms on infra-nilmanifolds modeled on free nilpotent Lie groups is twofold.

As explained in the introduction, a necessary condition to describe which infra-nilmanifolds in a certain class admit an Anosov diffeomorphisms is to understand the situation for the nilmanifolds of this class. A result of S.G. Dani in [23] shows that a nilmanifold modeled on a free nilpotent Lie group on n generators and of nilpotency class c admits an Anosov diffeomorphism if and only if $n > c$. So for free nilpotent Lie groups, the necessary conditions is fulfilled.

Note that the main theorem of this chapter indeed implies this result of [23]. If $N\backslash G_{n,c}$ is a nilmanifold, then the abelianized rational holonomy representation $\bar{\rho}$ is trivial since the holonomy group F is trivial. So $\bar{\rho}$ splits in exactly n irreducible representations over $\mathbb{Q}$ and these representations are also $\mathbb{R}$-irreducible. Theorem 10.1 then states that the infra-nilmanifold admits an Anosov diffeomorphism if and only if $1 > \frac{c}{n}$ or thus if and only if $n > c$.

The second reason for considering these infra-nilmanifolds is that they can be studied by techniques of representation for finite groups. We show how this can be done by introducing the abelianized rational holonomy representation.

Abelianized rational holonomy representation

Consider $\Gamma\backslash G$ an infra-nilmanifold with holonomy group F. From Theorem 3.36 we already know that the existence of an Anosov diffeomorphism depends only on the rational holonomy representation $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$ corresponding to the infra-nilmanifold $\Gamma\backslash G$.

The subgroup $\gamma_2(N^{\mathbb{Q}}) = [N^{\mathbb{Q}}, N^{\mathbb{Q}}]$ is a fully characteristic subgroup of $N^{\mathbb{Q}}$ and therefore every automorphism $\varphi : N^{\mathbb{Q}} \to N^{\mathbb{Q}}$ induces an automorphism

$$\bar{\varphi} : N^{\mathbb{Q}} / \gamma_2(N^{\mathbb{Q}}) \to N^{\mathbb{Q}} / \gamma_2(N^{\mathbb{Q}}).$$

The quotient group $N^{\mathbb{Q}} / \gamma_2(N^{\mathbb{Q}})$ is an abelian torsion-free radicable subgroup, so $N^{\mathbb{Q}} / \gamma_2(N^{\mathbb{Q}}) \simeq \mathbb{Q}^n$ for some n and thus the automorphism $\bar{\varphi} \in \mathrm{GL}(n, \mathbb{Q})$ under this isomorphism. The abelianized rational holonomy representation $\bar{\rho} : F \to \mathrm{GL}(n, \mathbb{Q})$ is the representation we get by considering this induced map of $\mathrm{GL}(n, \mathbb{Q})$ for every $\rho(f) \in \mathrm{Aut}(N^{\mathbb{Q}})$, so

$$\bar{\rho}(f) = \overline{\rho(f)}$$

for every $f \in F$.

In Section 6.5 we showed how we can study expanding maps and non-trivial self-covers on the infra-nilmanifold $\Gamma\backslash G$ by looking at the projection $\pi : \mathrm{Aut}(N^{\mathbb{Q}}) \to \mathrm{GL}(n, \mathbb{Q})$. Some information is lost when considering this projection and in particular the existence of an Anosov diffeomorphisms is not determined by the linear algebraic $\mathbb{Q}$-group we find in this way, see Example 6.31. When restricting to the case of free nilpotent Lie algebras though, there is no loss of information.

A matrix is called c-hyperbolic if its eigenvalues are c-hyperbolic algebraic integers. The following result translates the existence of an Anosov diffeomorphism into a question of representation theory for finite groups.

Theorem 10.2. *Let Γ be an almost-Bieberbach group modeled on a free nilpotent Lie group $G_{n,c}$ and denote the abelianized rational holonomy representation by $\bar{\rho} : F \to \mathrm{GL}(n, Q)$. Then the following statements are equivalent.*

$$\Gamma\backslash G_{n,c} \textit{ admits an Anosov diffeomorphism.}$$
$$\Updownarrow$$
$$\textit{There exists a } c\textit{-hyperbolic, integer-like automorphism } \psi \in \mathrm{GL}(n, \mathbb{Q})$$
$$\textit{which commutes with every element of } \rho(F).$$

We give a proof of this result to show what the crucial steps are. This is important to consider possible generalizations as in Section 10.5.

Proof. Write $\mathfrak{n}^{\mathbb{Q}}$ for the rational Lie algebra corresponding to the radicable hull $N^{\mathbb{Q}}$ of the Fitting subgroup of Γ. This Lie algebra will be a free rational nilpotent Lie algebra on n generators and of nilpotency class c.

$\boxed{\Downarrow}$ If φ is an Anosov automorphism commuting with every elements of $\rho(F)$, then the induced map $\bar{\varphi} : \mathfrak{n}^{\mathbb{Q}} / \gamma_2(\mathfrak{n}^{\mathbb{Q}}) \to \mathfrak{n}^{\mathbb{Q}} / \gamma_2(\mathfrak{n}^{\mathbb{Q}})$ commutes with every element of $\bar{\rho}(F)$. Denote by λ_i the eigenvalues of $\bar{\varphi}$. Every eigenvalue of φ is a k-fold product $\lambda_{i_1} \ldots \lambda_{i_k}$ and all of these k-fold products occur as an eigenvalue, except for possibly the eigenvalue λ_i^2. So since φ is hyperbolic, the induced map $\bar{\varphi}$ is c-hyperbolic.

$\boxed{\Uparrow}$ Let $\psi \in \mathrm{GL}(n, \mathbb{Q})$ be c-hyperbolic and integer-like. Since $\gamma_2(\mathfrak{n}^{\mathbb{Q}})$ is invariant under $\bar{\rho}(F)$, there exists a complementary subspace $\mathfrak{n}_1^{\mathbb{Q}}$ which is invariant under $\bar{\rho}(F)$ and such that $\mathfrak{n}^{\mathbb{Q}} = \mathfrak{n}_1^{\mathbb{Q}} \oplus \gamma_2(\mathfrak{n}^{\mathbb{Q}})$. Consider ψ as an automorphism of the rational vector space $\mathfrak{n}_1^{\mathbb{Q}}$ and take φ the automorphism of $\mathfrak{n}^{\mathbb{Q}}$ induced by ψ. By construction, φ commutes with every element of $\rho(F)$. The automorphism φ is integer-like since all its eigenvalues are algebraic units, see also the proof of Theorem 9.14. Since ψ is c-hyperbolic, φ is hyperbolic as in the proof of the other implication. This implies that the manifold $\Gamma\backslash G_{n,c}$ admits an Anosov diffeomorphism. $\square$

In the proof two properties of the Lie group $G_{n,c}$ are crucial, each needed in a different implication. The first is that for every $1 \leq k \leq c$, every k-fold product of eigenvalues λ_i of the induced automorphism $\bar{\varphi}$ occurs as an eigenvalue of φ, except possibly for some squares λ_i^2. Note that these squares do not form a problem since $|\lambda_i^2| = |\lambda_i|^2$. The second property crucial for the proof is that every automorphism in $\mathrm{GL}(n,\mathbb{Q})$ induces an automorphism of the free Lie group $G_{n,c}$ or thus that the map $\pi : \mathrm{Aut}(N^{\mathbb{Q}}) \to \mathrm{GL}(n,\mathbb{Q})$ is surjective.

Equivalent statement for representations

Theorem 10.2 shows that we have to find c-hyperbolic elements of $\mathrm{GL}(n,\mathbb{Q})$ which commute with representations $\rho : H \to \mathrm{GL}(n,\mathbb{Q})$ of arbitrary finite groups H. We use the notation H for finite groups here since the groups are not always coming from a infra-nilmanifold Γ and since F is used as a field extension in the following paragraphs. By decomposing this representation ρ into its $\mathbb{Q}$-irreducible components and grouping the components which are equivalent over $\mathbb{Q}$, we can assume that ρ is a direct sum $\rho_1 \oplus \ldots \oplus \rho_1$ where ρ_1 is an $\mathbb{Q}$-irreducible representation. The following theorem is therefore equivalent to Theorem 10.1.

Theorem 10.3. *Let H be a finite group and $\rho : H \to \mathrm{GL}(n,\mathbb{Q})$ a $\mathbb{Q}$-irreducible representation. Then the following are equivalent.*

There exists a c-hyperbolic, integer-like matrix $C \in \mathrm{GL}(mn,\mathbb{Q})$
which commutes with $m\rho = \underbrace{\rho \oplus \rho \oplus \cdots \oplus \rho}_{m \text{ times}}$.

$$\Updownarrow$$

ρ splits in strictly more than $\frac{c}{m}$ components
when seen as a representation over $\mathbb{R}$.

It might be surprising at first that the criterion in Theorem 10.1 depends only on how the representation splits over $\mathbb{R}$ and not on how it splits over $\mathbb{C}$. This is a consequence of the fact that if an $\mathbb{R}$-irreducible representation splits over $\mathbb{C}$, it can be decomposed into exactly two subrepresentations which can be assumed to be complex conjugates. Every matrix $A \in \mathrm{GL}(2n,\mathbb{C})$ commuting with this representation can then also be written in the form

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & \bar{A}_0 \end{pmatrix}$$

for some $A_0 \in \mathrm{GL}(n,\mathbb{C})$. Since for every complex number $\lambda \in \mathbb{C}$ we have that $|\lambda| = |\overline{\lambda}|$, the matrix A is c-hyperbolic if and only if the matrix A_0 is c-hyperbolic. This shows that the splitting over $\mathbb{C}$ doesn't give us more information than the splitting over the reals $\mathbb{R}$.

Idea of the proof for abelian groups

To end this introductory section, we sketch the proof of Theorem 10.3 in the case where the finite group H is abelian. This is also, up to some simplifications, the proof as it was presented in [40, Theorem 2.3.]. The proof for general groups H follows the same outline.

The following fact is a well-known result about division rings, see [60].

Theorem 10.4. *Every finite, abelian subgroup of the multiplicative group of a division ring is cyclic.*

Let $\rho : H \to \mathrm{GL}(n, \mathbb{Q})$ be a $\mathbb{Q}$-irreducible representation. The algebra of H-morphisms of $\mathbb{Q}^n$ forms a division ring by the Lemma of Schur. Therefore the group $\rho(H)$ is cyclic as a finite abelian subgroup of the multiplicative group of this division ring. So by taking a generator of $\rho(H)$, it suffices to check Theorem 10.3 for matrices of finite order.

Let A be a generator of the image of $\rho(H)$ with characteristic polynomial $p(X)$. Denote by E the splitting field of $p(X)$, which is always a Galois extension of $\mathbb{Q}$. If v_0 is an eigenvector for the eigenvalue λ_0, then a basis of eigenvectors for A is given by $\sigma(v_0)$ for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$. This basis of eigenvectors, which are all Galois conjugates, gives us information about the matrices commuting with A, since they are also diagonal matrices in this basis. Vice versa, by taking diagonal matrices in this basis of eigenvectors which are 'symmetric' under the action of the Galois group, we can construct matrices commuting with A. Being 'symmetric' under the action of the Galois group means that the eigenvalue corresponding to $\sigma(v_0)$ should be equal to $\sigma(\lambda)$ where λ is the eigenvalue of the eigenvector v_0. A similar analysis of direct sums of $\mathbb{Q}$-irreducible representations of abelian groups then implies Theorem 10.3, see [40].

In general, a representation $\rho : H \to \mathrm{GL}(n, \mathbb{Q})$ does not have a basis consisting of eigenvectors for every $\rho(h)$, although it does have a decomposition in $\mathbb{C}$-irreducible representations. To exploit the same ideas as in the abelian case, we need to construct a decomposition of ρ into its $\mathbb{C}$-irreducible components which is also given by Galois conjugates.

One problem is that there is not a unique splitting field like in the abelian case, where we had the splitting field of the characteristic polynomial. Another problem is that the fields over which ρ decomposes into its $\mathbb{C}$-irreducible components are not always Galois over $\mathbb{Q}$. So we have to find an equivalent statement for arbitrary number fields and translate the 'symmetric' property we used for Galois extensions. Finally, the $\mathbb{C}$-irreducible components are not always 1-dimensional as in the abelian case and we will need more information about their dimensions as well.

In the following section, we address these three problems independently. This allows us to prove the main theorem in Section 10.3.

10.2 Decomposition over minimal splitting fields

In the introduction, we sketched the proof of Theorem 10.3 in the specific case of finite abelian groups. In this section we construct the main tools for generalizing this proof to arbitrary groups. The main difference is that the basis of eigenvectors, which are Galois conjugates, is replaced by the decomposition of the representation into its $\mathbb{C}$-irreducible components.

There are three problems that occur during this generalization to arbitrary $\mathbb{Q}$-irreducible representation. First, we have to describe over which extension fields of $\mathbb{Q}$ the representation splits and such fields are called minimal splitting fields. Next, given such a splitting field for the representation, we are looking for a similar decomposition into Galois conjugates as in the abelian case. Finally, in the abelian case it was obvious that all $\mathbb{R}$-irreducible components were of the same dimension. For general representation we need the Frobenius-Schur indicator to conclude this statement.

All the groups H we consider in this section are finite. In the next sections, we discuss minimal splitting fields as introduced in Section 5.1 and study how a $\mathbb{Q}$-irreducible representation splits over such a minimal splitting field.

10.2.1 Characters afforded by real representations

For Theorem 10.3, we are particularly interested in the real representations of H or equivalently in the characters which are afforded by a real representation. The Frobenius-Schur indicator completely describes whether a $\mathbb{C}$-irreducible character has a real representation or not.

Let H be a finite group and $\mathrm{Irr}(H)$ the set of $\mathbb{C}$-irreducible characters of H. The Frobenius-Schur indicator $\nu_2(\chi)$ of an irreducible character $\chi \in \mathrm{Irr}(H)$ is defined as the complex number

$$\nu_2(\chi) = \frac{1}{|H|} \sum_{h \in H} \chi(h^2).$$

From [68, page 58] or [94, Prop. 39, page 109] it follows that $\nu_2(\chi)$ can only have the values $1, 0$ and -1 and this value completely determines whether χ is afforded by a real representation or not.

Theorem 10.5. *Let $\chi \in \mathrm{Irr}(H)$ be an irreducible character. Then we have the following possibilities for the Frobenius-Schur indicator $\nu_2(\chi)$.*

- $\nu_2(\chi) = 0$ *if and only if χ is not real (i.e. $\exists h \in h$ such that $\chi(h) \notin \mathbb{R}$).*
- $\nu_2(\chi) = 1$ *if and only if χ is afforded by a real representation.*
- $\nu_2(\chi) = -1$ *if and only if χ is real but not afforded by a real representation.*

From the definition it follows that the Frobenius-Schur indicator ν_2 is identical for all Galois conjugates χ^σ of an irreducible character.

Lemma 10.6. *Let $\chi \in \mathrm{Irr}(G)$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi), \mathbb{Q})$, then we have that $\nu_2(\chi) = \nu_2(\chi^\sigma)$.*

Proof. By using the formula for ν_2, we have

$$\begin{aligned} \nu_2(\chi^\sigma) = \frac{1}{|G|}\sum_{g\in G}\chi^\sigma(g^2) &= \frac{1}{|G|}\sum_{g\in G}\sigma\left(\chi(g^2)\right) \\ &= \sigma\left(\frac{1}{|G|}\sum_{g\in G}\chi(g^2)\right) \\ &= \sigma(\nu_2(\chi)) = \nu_2(\chi) \end{aligned}$$

since $\nu_2(\chi) \in \mathbb{Z}$. □

Since all $\mathbb{C}$-irreducible components of a $\mathbb{Q}$-irreducible representation are Galois conjugates, this implies that the Frobenius-Schur indicator is well-defined for $\mathbb{Q}$-irreducible characters. This observation has the following important consequence about the dimension of $\mathbb{R}$-irreducible components.

Theorem 10.7. *All $\mathbb{R}$-irreducible components of an $\mathbb{Q}$-irreducible representation have the same dimension.*

Proof. Let χ be the character of a $\mathbb{C}$-irreducible component of the $\mathbb{Q}$-irreducible representation ρ.

If $\nu_2(\chi) = 1$, then all irreducible components of ρ are afforded by a real representation, since they are all of the form χ^σ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi), \mathbb{Q})$ ([68, Lemma 9.21 (c)]). Thus the $\mathbb{R}$-irreducible components of ρ are equal to the $\mathbb{C}$-irreducible components and they all have the same dimension

$$\chi^\sigma(e_G) = \sigma(\chi(e_G)) = \chi(e_G).$$

If $\nu_2(\chi) \neq 1$, then none of the irreducible components is afforded by a real representation and thus every $\mathbb{R}$-irreducible component has the same dimension $\chi(e_G) + \overline{\chi}(e_G) = 2\chi(e_G)$. □

10.2.2 Decomposition over a minimal splitting field

If H is a finite abelian group, then every $\mathbb{Q}$-irreducible representation has a basis of eigenvectors which are Galois conjugates. With this basis, it is easy to construct rational matrices commuting with every element of the image of this representation. This sections generalizes this statement to arbitrary groups.

To describe the main result of this section, we first have to introduce permutation matrices. Let $\pi \in S_n$ be a permutation, then there exists a permutation matrix $P_\pi \in \mathrm{GL}(n, \mathbb{Z})$, which is defined by

$$(P_\pi)_{ij} = \begin{cases} 1 & j = \pi(i) \\ 0 & \text{otherwise.} \end{cases}$$

So the linear map given by the matrix P_π induces the permutation π^{-1} on the standard basis e_i and this property uniquely defines the matrix P_π. Using this observation, it follows that for every $\pi_1, \pi_2 \in S_n$ we have

$$P_{\pi_1} P_{\pi_2} = P_{\pi_2 \pi_1}$$

since both induce the permutation $\pi_1^{-1}\pi_2^{-1} = (\pi_2\pi_1)^{-1}$ on the standard basis. In particular we have that $P_\pi^{-1} = P_{\pi^{-1}}$.

Let M be a matrix in $GL(n, \mathbb{C})$, then $P_\pi M$ is the matrix M where the rows are permuted according to π, so the i-th row of $P_\pi M$ is the $\pi(i)$-th row of M. If we write

$$M = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix},$$

as a column of row vectors, then an equivalent way of saying this is that

$$P_\pi M = \begin{pmatrix} m_{\pi(1)} \\ m_{\pi(2)} \\ \vdots \\ m_{\pi(n)} \end{pmatrix}.$$

In the same way, MP_π is the matrix M where the columns are permuted according to π^{-1}. This last property can easily be checked by observing that the transpose of P_π satisfies

$$(P_\pi)^T = P_{\pi^{-1}}$$

and thus

$$MP_\pi = (P_{\pi^{-1}} M^T)^T.$$

Recall that the Kronecker product $A \otimes B$ of two matrices A and B is defined as the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1l}B \\ a_{21}B & a_{22}B & \ldots & a_{2l}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}B & a_{k2}B & \ldots & a_{kl}B \end{pmatrix}.$$

The Kronecker product $P_\pi \otimes \mathbb{1}_k \in \mathrm{GL}(kn, \mathbb{Z})$ will be denoted as $P_\pi^{\otimes k}$. If we write a matrix $M \in GL(kn, \mathbb{C})$ as a column of $k \times kn$ matrices

$$M = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix},$$

then similarly as above $P_\pi^{\otimes k} M$ is the matrix M with the block matrices m_i permuted according to the permutation π.

Let L be a Galois extension over $\mathbb{Q}$ of finite degree. Fix a subfield $K = \mathbb{Q}(\theta) \subseteq L$, which is not necessarily Galois over $\mathbb{Q}$. Denote by $\sigma_1, \ldots, \sigma_n$ the n distinct monomorphisms $K \to \mathbb{C}$. The monomorphism σ_i is completely determined by the image of θ and denote this image as $\theta_i = \sigma_i(\theta)$ which is a root of the characteristic polynomial of θ. Since E is Galois over $\mathbb{Q}$, it contains all roots of the characteristic polynomial of θ, so $\theta_i \in E$. This implies that the field $\sigma_i(K) \subseteq E$ for all i. Fix an element $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ of the Galois group. For evyer $i \in \{1, 2, \ldots, n\}$, we have that $\sigma \circ \sigma_i : K \to E \subseteq \mathbb{C}$ is again a monomorphism of K in $\mathbb{C}$. Hence σ induces a permutation on $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$. Associated to this is a permutation $\pi_\sigma \in S_n$ which is determined by

$$\sigma \circ \sigma_i = \sigma_{\pi_\sigma(i)}.$$

We let $P_\sigma \in \mathrm{GL}(n, \mathbb{Z})$ denote the corresponding permutation matrix.

Let ρ be a $\mathbb{Q}$-irreducible representation of a finite abelian group H with E the minimal splitting field of ρ. If we write $\mathrm{Gal}(E, \mathbb{Q}) = \{\sigma_1, \dots, \sigma_n\}$ then there exists a eigenvector $v_0 \in E^n$ of eigenvalue λ_0 such that with respect to the basis $\{v_1 = \sigma_1(v_0), v_2 = \sigma_2(v_0), \dots, v_n = \sigma_n(v_0)\}$, the generator of the image of ρ is given by the matrix

$$\begin{pmatrix} \sigma_1(\lambda_0) & 0 & \dots & 0 \\ 0 & \sigma_2(\lambda_0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n(\lambda_0) \end{pmatrix}.$$

The matrix of change of basis Q, given by

$$Q = \begin{pmatrix} \sigma_1(v_0) & \sigma_2(v_0) & \dots & \sigma_n(v_0) \end{pmatrix}$$

obviously satisfies the relation

$$\sigma(Q) = QP_{\sigma^{-1}}^{\otimes k}.$$

This shows that the following theorem is indeed a generalization of the existence of a basis of Galois conjugates to arbitrary groups.

Theorem 10.8. *Let $\chi \in \mathrm{Irr}(H)$ be the character of an irreducible component of a $\mathbb{Q}$-irreducible representation ρ. Let $\mathbb{Q} \subseteq K$ be a field extension of minimal degree such that χ is afforded by a K-representation, say ρ_0 and assume that k is the dimension of this representation. If $[K : \mathbb{Q}] = n$ and $\sigma_1, \dots, \sigma_n$ are the n distinct monomorphisms $K \to \mathbb{C}$ and if L is any field extension $\mathbb{Q} \subseteq K \subseteq L$ such that L is Galois over $\mathbb{Q}$, then there exist a $Q \in \mathrm{GL}(kn, L)$ such that the following conditions hold:*

(1) $$Q^{-1}\rho Q = \begin{pmatrix} \sigma_1(\rho_0) & 0 & \dots & 0 \\ 0 & \sigma_2(\rho_0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n(\rho_0) \end{pmatrix}.$$

(2) *For all $\sigma \in \mathrm{Gal}(L, \mathbb{Q})$, we have that*

$$\sigma(Q) = QP_{\sigma^{-1}}^{\otimes k}.$$

Proof. Let $\rho_0 : H \to GL(k, K)$ be an irreducible representation with character χ. Consider the representations $\rho^i = \sigma_i(\rho_0) : H \to \mathrm{GL}(k, \sigma_i(K))$ with character χ^{σ_i}. We construct a new representation

$$\tilde{\rho} : H \to \mathrm{GL}(kn, L)$$
$$g \mapsto \begin{pmatrix} \rho_g^1 & 0 & \dots & 0 \\ 0 & \rho_g^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_g^n \end{pmatrix},$$

which was already used in the formulation of the theorem.

Since K is a finite field extension of $\mathbb{Q}$, we know that $K = \mathbb{Q}(\theta)$ for some algebraic number θ. Look at the matrix

$$R = \begin{pmatrix} \sigma_1(1) & \sigma_1(\theta) & \dots & \sigma_1(\theta^{n-1}) \\ \sigma_2(1) & \sigma_2(\theta) & \dots & \sigma_2(\theta^{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n(1) & \sigma_n(\theta) & \dots & \sigma_n(\theta^{n-1}) \end{pmatrix} \otimes \mathbb{1}_k,$$

then it follows by construction that $\sigma(R) = P_\sigma^{\otimes k} R$ for every $\sigma \in \mathrm{Gal}(L, \mathbb{Q})$. Note that R is the Kronecker product of a Vandermonde matrix with the identity and thus its determinant can easily be computed and is different from 0. So R is invertible and denote the inverse of R by Q. By applying $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ to the relation $RQ = \mathbb{1}_{kn}$, we find that

$$\sigma(Q) = \sigma(R)^{-1} = R^{-1}(P_\sigma^{\otimes k})^{-1} = QP_{\sigma^{-1}}^{\otimes k}$$

and thus the matrix Q satisfies condition 2 of the theorem.

First we show that $Q\tilde{\rho}_g R \in \mathrm{GL}(nk, \mathbb{Q})$ for all $g \in H$. Equivalently, we have to show that $\sigma(P\tilde{\rho}_g Q) = P\tilde{\rho}_g Q$ for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$. We compute that

$$\sigma(Q\tilde{\rho}_g R) = \sigma(Q)\sigma(\tilde{\rho}_g)\sigma(R) = QP_{\sigma^{-1}}^{\otimes k}\sigma(\tilde{\rho}_g)P_\sigma^{\otimes k}R.$$

It is easy to see that $\sigma(\tilde{\rho}_g) = P_\sigma^{\otimes k}\tilde{\rho}_g P_{\sigma^{-1}}^{\otimes k}$ and thus the conclusion follows. So it suffices to show that the representations $Q\tilde{\rho}R$ and ρ are equivalent over $\mathbb{Q}$.

Note that $\tilde{\rho}$ and thus also $Q\tilde{\rho}R$ have character $\chi^{\sigma_1} + \ldots + \chi^{\sigma_n}$. So the representations $Q\tilde{\rho}R$ and ρ have a common $\mathbb{C}$-irreducible factor, namely ρ_0, corresponding to the character χ. Also, it follows from the discussion above about the characters of $\mathbb{Q}$-irreducible representations that both have the same dimension, namely kn. As a consequence of [68, Corollary 9.7], we have that $Q\tilde{\rho}R$ has a $\mathbb{Q}$-irreducible component which is equivalent with ρ over $\mathbb{Q}$. Because they have the same dimension, this ends the proof of the theorem. $\square$

In the statement of Theorem 10.8 the matrix Q satisfies the relation

$$\sigma(Q) = QP_{\sigma^{-1}}^{\otimes k}$$

for all $\sigma \in \mathrm{Gal}(L, \mathbb{Q})$. This condition is interesting since we can easily construct matrices commuting with the original representation and that have coefficients in $\mathbb{Q}$.

Proposition 10.9. *Let $K \supseteq \mathbb{Q}$ be a field extension of degree n and $\sigma_1, \ldots, \sigma_n$ the distinct monomorphisms from K to $\mathbb{C}$. Let C_0 be any matrix with coefficients in K and look at the matrix*

$$C = \begin{pmatrix} \sigma_1(C_0) & 0 & \cdots & 0 \\ 0 & \sigma_2(C_0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n(C_0) \end{pmatrix}.$$

Let L be any field extension of K which is Galois over $\mathbb{Q}$ and Q an invertible matrix with entries in L which satisfies

$$\sigma(Q) = QP_{\sigma^{-1}}^{\otimes k}$$

for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$, where we use notations as above. Then QCQ^{-1} is a matrix with coefficients in $\mathbb{Q}$.

Proof. Just as in Theorem 10.8, the matrix C satisfies

$$\sigma(C) = P_{\sigma}^{\otimes k} C P_{\sigma^{-1}}^{\otimes k}$$

for every $\sigma \in \mathrm{Gal}(L, \mathbb{Q})$. By applying $\sigma \in \mathrm{Gal}(L, \mathbb{Q})$ to the relation $Q^{-1}Q = \mathbb{1}$, we also get that

$$\sigma(Q^{-1}) = P_{\sigma}^{\otimes k} Q^{-1}.$$

Take $\sigma \in \mathrm{Gal}(L, \mathbb{Q})$ arbitrary, then

$$\sigma(QCQ^{-1}) = QCQ^{-1}$$

and therefore QCQ^{-1} has coefficients in $\mathbb{Q}$. $\square$

Remark 10.10. Note that the matrix C of the previous proposition doesn't depend on the choice of Galois extension L. Also the construction of the matrix Q in the proof of Theorem 10.8 didn't depend on the field L. It was only used to embed the set of monomorphisms σ_i into a nice group structure, such that they are easier to handle. In the rest of this chapter we will ignore the use of the Galois extension L.

10.2.3 Existence of real minimal splitting fields

In the previous section, we showed how a $\mathbb{Q}$-irreducible representation splits over any minimal splitting field. For a $\mathbb{Q}$-irreducible representation ρ of an abelian group, there is always a canonical choice of a minimal field extension $F \supseteq \mathbb{Q}$ such that the representation is completely reducible over F, i.e. is diagonalizable over F. If $p(X) \in \mathbb{Q}[X]$ is the characteristic polynomial of a generator of the image of ρ, then the field K is equal to the splitting field of $p(X)$ over $\mathbb{Q}$.

Contrary to the situation of abelian groups, this minimal splitting field is in general not unique. In this section, we construct real minimal splitting fields in the case where the representation is completely reducible over $\mathbb{R}$.

First we give an example showing that minimal splitting fields are not unique for general groups.

Example 10.11. Consider the unique two-dimensional irreducible representation of the quaternion group Q_8. Take complex numbers $\alpha, \beta \in \mathbb{C}$ with $\alpha^2 + \beta^2 = -1$ and look at the representation ρ given by

$$\rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho(j) = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}.$$

It's an easy exercise to check that this indeed defines a representation of Q_8. This shows that every field of the form $\mathbb{Q}(\sqrt{-1-\alpha^2})$ with $\alpha \in \mathbb{Q}$ is a minimal splitting field for this representation. These fields do have in common that they are not real. In fact it's an exercise to show that every minimal splitting field for this character is totally imaginary.

So this example seems to suggest that the minimal splitting field is imaginary if and only if the representation is not affordable over the reals. In general it's not true that if a character χ is afforded by an L-representation for a field $L \supseteq \mathbb{Q}$, that there is also a subfield $K \subseteq L$ such that K has minimal degree and χ is afforded by a K-representation. Therefore we cannot directly conclude that every real representation has a real minimal splitting field. In fact, in Theorem 10.15 we will show that even real representation sometimes have minimal splitting fields which are not real.

Let $\chi \in \mathrm{Irr}(H)$ be a character which is afforded by a real representation. If $m_{\mathbb{Q}}(\chi) = 1$, or said differently, if $\mathbb{Q}(\chi)$ is a splitting field for ρ, then of course every minimal splitting field of χ is real. By the Brauer-Speiser Theorem, see for example [68, page 171], we know that $m_{\mathbb{Q}}(\chi) \leq 2$ and thus the only situation left to check is the one with $m_{\mathbb{Q}}(\chi) = 2$.

Real minimal splitting fields

First, we characterize the existence of a real minimal splitting field in terms of H-isomorphisms. Since this H-isomorphism is defined over the field K, this gives us an easier way to check if there is a real minimal splitting field.

Lemma 10.12. *Let χ be an irreducible real valued character of H with $m_{\mathbb{Q}}(\chi) = 2$. Let $K = \mathbb{Q}(\chi) \subseteq \mathbb{R}$ and $\rho : H \to \mathrm{GL}(2n, K)$ a representation with character 2χ. Then the following statements are equivalent:*

(1) *There exists a minimal splitting field $L \subseteq \mathbb{R}$.*

(2) *There exists an H-isomorphism $f : K^{2n} \to K^{2n}$ with $f^2 = \kappa \mathbb{1}_{K^{2n}}$, $\kappa > 0$ and $\sqrt{\kappa} \notin K$.*

Proof. First we show that (1) implies (2). Let F be such a minimal splitting field, so $F = K(\sqrt{d})$ with $d \notin K^2$, $d > 0$ and there exists an F-representation ρ_0 which affords the character χ. Denote by σ the nontrivial element of $\mathrm{Gal}(F, K)$. By using the same techniques as in the proof of Theorem 10.8, we can show that there exists a matrix P such that

$$P^{-1}\rho P = \begin{pmatrix} \rho_0 & 0 \\ 0 & \sigma(\rho_0) \end{pmatrix},$$

with $\sigma(P) = PK_\sigma^{\otimes n}$. Now consider the matrix

$$C = \begin{pmatrix} \sqrt{d}\mathbb{1}_n & 0 \\ 0 & \sigma(\sqrt{d})\mathbb{1}_n \end{pmatrix},$$

then it follows, just like in Proposition 10.9, that PCP^{-1} has coefficients in K and commutes with the representation ρ. It's easy to check that the linear map f induced by this matrix satisfies all the wanted properties.

Next we prove that the existence of f gives us a minimal splitting field. Take $F = K(\sqrt{\kappa})$ and σ as before. Consider the representation ρ as an F-representation by extending the scalars, in the same way the map f is also an G-isomorphism over F. Since $f^2 = \kappa \mathbb{1}_{F^{2n}}$, we know that f can only have two different eigenvalues, namely $\sqrt{\kappa}$ and $-\sqrt{\kappa}$. Each of those eigenvalues occurs, since if v is an eigenvector for $\sqrt{\kappa}$, then $\sigma(v)$ is an eigenvector for $\sigma(\sqrt{\kappa}) = -\sqrt{\kappa}$ and vice versa. Now the splitting of F^{2n} into the eigenspaces of f gives us a decomposition of F^{2n} into two G-invariant subspaces and thus F is a splitting field. Because $F \subseteq \mathbb{R}$, this ends our proof. □

We use this lemma to prove the existence of a real minimal splitting field.

Theorem 10.13. *Let $\chi \in \mathrm{Irr}(G)$ be an irreducible character which is afforded by a real representation. Then there exists a real minimal splitting field for χ.*

Proof. We use some ideas of the proof of [94, Theorem 31], but in our case, we work over a finite field extension of $\mathbb{Q}$ instead of over $\mathbb{C}$. Let $K = \mathbb{Q}(\chi)$, then we know that K is a real field extension of $\mathbb{Q}$. As mentioned above, we only have to check the case where $m_{\mathbb{Q}}(\chi) = 2$ because of the Brauer-Speiser Theorem. So there exists a field $F \supseteq K$ such that χ is afforded by a F-representation ρ and $[F : K] = 2$. If $F \subseteq \mathbb{R}$, there is nothing to prove, so we can assume that $F = K(\theta)$ with $\theta = \sqrt{d}$, $d \in K$ and $d < 0$.

Recall that ρ also induces a representation on the dual vector space V^*, namely $\rho^* : G \to GL(V^*)$ with

$$\rho^*_g(\varphi) = \varphi \circ (\rho_g)^{-1} = \varphi \circ \rho_{g^{-1}} \qquad \forall \varphi \in V^*.$$

Since the values of χ lie in $\mathbb{R}$, the character of ρ^* is equal to χ. So the representations ρ and ρ^* are equivalent and there exists a G-isomorphism $f : V \to V^*$. Now look at the map

$$B : V \times V \to F : (v, w) \mapsto f(v)(w).$$

It is easy to see that B is a nondegenerate bilinear form on V. Since f is a G-isomorphism, B is clearly G-invariant, i.e. $B(\rho_g(v), \rho_g(w)) = B(v, w)$ for all $g \in G$, $v, w \in V$. This construction gives us an isomorphism between the space of G-morphisms from V to V^* and the space of G-invariant bilinear forms on V. Since V and V^* are irreducible, the space of G-morphisms between them has dimension 1 . Hence, this G-invariant isomorphism f is unique up to scalar multiplication.

We can also consider the spaces $V_{\mathbb{C}} = \mathbb{C} \otimes_F V$ and $(V_{\mathbb{C}})^* = (V^*)_{\mathbb{C}} = \mathbb{C} \otimes_F V^*$ and the induced representations $\rho_{\mathbb{C}} : G \to \mathrm{GL}(V_{\mathbb{C}})$ and $\rho^*_{\mathbb{C}} : G \to \mathrm{GL}(V^*_{\mathbb{C}})$. Then f extends to a G-isomorphism $f_{\mathbb{C}} : V_{\mathbb{C}} \to V^*_{\mathbb{C}}$ and the bilinear form B extends to $B_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C} : (v, w) \mapsto f_{\mathbb{C}}(v)(w)$. Since χ is afforded by a $\mathbb{R}$-representation, there exists also a nondegenerate symmetric bilinear form over $\mathbb{C}$ which is invariant under G by [94, Theorem 31]. Because of the uniqueness of nondegenerate G-invariant bilinear forms up to scalar multiplication, it now follows that $B_{\mathbb{C}}$, and hence also B, must be symmetric.

Now choose a G-invariant, positive definite, hermitian scalar product

$$\langle\ ,\ \rangle : V \times V \to F.$$

For every $v \in V$, there exists a unique $\psi(v) \in V$ such that $B(w, v) = \langle w, \psi(v)\rangle$ for all $w \in V$. An easy computation then shows that ψ is bijective and antilinear, i.e. $\psi(\lambda v) = \overline{\lambda}\psi(v)$ for all $\lambda \in F$. Also, ψ is G-invariant because both B and $\langle\ ,\ \rangle$ are G-invariant. So ψ^2 is a G-automorphism of V and thus $\psi^2 = \mu \mathbb{1}_V$ for some $\mu \in F$. By looking at V as a vector space over K, the map ψ becomes linear (since $K \subseteq \mathbb{R}$). Therefore, it is sufficient to show that $\mu \in K, \mu > 0$ and $\sqrt{\mu} \notin K$ because of Lemma 10.12.

For any $v, w \in V$ we have that

$$\langle v, \psi(w)\rangle = B(v, w) = B(w, v) = \langle w, \psi(v)\rangle = \overline{\langle \psi(v), w\rangle}.$$

Using this identity, we find for all $v, w \in V$ that

$$\overline{\mu}\langle v, w\rangle = \langle v, \mu w\rangle = \langle v, \psi^2(w)\rangle = \overline{\langle \psi(v), \psi(w)\rangle} = \langle \psi^2(v), w\rangle = \mu\langle v, w\rangle$$

and thus $\mu \in \mathbb{R} \cap F = K$. From the same computation with $v = w$, we also have that $\mu\langle v, v\rangle = \langle \psi(v), \psi(v)\rangle$ and $\mu > 0$ because $\langle\ ,\ \rangle$ is positive definite. It's easy to check that $\sqrt{\mu} \notin K$ because χ cannot be afforded by a K-representation. This ends the proof. $\square$

Minimal splitting fields which are not real

For completeness we discuss the existence of minimal splitting fields which are not real, although the character is afforded by a real representation. This shows the importance of Theorem 10.13, since not every minimal splitting field of such a representation is real.

We start with a similar characterization for the existence of minimal splitting fields that are not real as in Lemma 10.12.

Lemma 10.14. *Let χ be an irreducible character of H with Schur index 2. Let $K = \mathbb{Q}(\chi) \subseteq \mathbb{R}$ and $\rho : H \to \mathrm{GL}(2n, K)$ a representation with character 2χ. Then the following statements are equivalent.*

(1) *There exists a non-real minimal splitting field.*

(2) *There exists an H-isomorphism $f : K^{2n} \to K^{2n}$ with $f^2 = \kappa \mathbb{1}_{K^{2n}}$ and $\kappa < 0$.*

The proof is almost identical as in Lemma 10.12.

By combining this lemma with Lemma 10.12 and Theorem 10.13, this gives us the following result.

Theorem 10.15. *Let χ be an irreducible character of G with Schur index 2 and which is afforded by a real representation. Then χ has both a real and a non-real minimal splitting field.*

Proof. The existence of a real minimal splitting field was given in Theorem 10.13. We now show that if χ has a real minimal splitting field, it also has a non-real minimal splitting field. Let $F = K(\theta)$ be a real minimal splitting field, with $\theta = \sqrt{d}$ and $d > 0$, and take $\sigma \in \mathrm{Gal}(F, K)$ as before. Then F^{2n} can be decomposed into two vector spaces $V \oplus \sigma(V)$ which are G-invariant. If we look at the representation ρ as a matrix representation, then we find that

$$\rho = \begin{pmatrix} \rho_0 & 0 \\ 0 & \sigma(\rho_0) \end{pmatrix}$$

for some representation $\rho_0 : G \to GL(n, F)$. Since $\sigma(\chi) = \chi$, we know that there exists a matrix $J \in \mathrm{GL}(n, F)$ such that $\rho_0 J = J\sigma(\rho_0)$. Since we also have that $\sigma(J)\rho_0 = \sigma(\rho_0)J$ and ρ_0 is absolutely irreducible, we know that $\sigma(J)J = \kappa \mathbb{1}_{F^n}$ for some $\kappa \in F$. Since the trace of $\sigma(J)J$ is invariant under σ, we know that $\sigma(\kappa) = \kappa$ and thus $\kappa \in K$. Note that also $J\sigma(J) = \kappa \mathbb{1}_{F^n}$. By replacing J with θJ if necessary, we can assume that $\kappa < 0$. Now take the G-isomorphism

$$f = \begin{pmatrix} 0 & \sigma(J) \\ J & 0 \end{pmatrix},$$

then f also induces a G-isomorphism on K^{2n}. It is easy to check that $f^2 = \kappa \mathbb{1}_{K^{2n}}$ and because of the previous proposition we conclude that χ has a non-real minimal splitting field. □

10.3 Proof of Theorem 10.1

In the previous part, we discussed how and over which fields $\mathbb{Q}$-irreducible representations of finite groups split. We now apply these results to prove the main theorem of this chapter.

Proof of $\Downarrow$:

If a matrix commutes with a representation, then its generalized eigenspaces are subrepresentation. By applying this fact in combination with Theorem 10.7 we get some information about the product of the eigenvalues. In particular, we have the following proposition as a natural consequence of Theorem 10.7.

Proposition 10.16. *Let $\rho : H \to GL(kr, \mathbb{Q})$ be a $\mathbb{Q}$-irreducible representation with r the dimension of every $\mathbb{R}$-irreducible component of ρ. If $C \in GL(krm, \mathbb{Q})$ commutes with $m\rho = \rho \oplus \rho \oplus \ldots \oplus \rho$ and $|\det(C)| = 1$, then C cannot be km-hyperbolic.*

Proof. By using the real Jordan canonical form of C (see e.g. [66, Theorem 3.4.5.]), we find an invertible matrix $A \in GL(krm, \mathbb{R})$ such that

$$ACA^{-1} = \begin{pmatrix} C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & C_l \end{pmatrix}$$

where C_i and C_j have distinct eigenvalues for $i \neq j$ and such that C_i either has one real eigenvalue or two complex conjugate eigenvalues. Let λ_i be one (of the at most two) eigenvalue(s) of C_i. So all λ_i are distinct by construction. By conjugating with the same matrix A, the representation $m\rho$ also splits over $\mathbb{R}$, since the generalized eigenspaces of C are obviously invariant under the representation $m\rho$. This implies that each C_i has dimension $k_i r$ for some $k_i \in \mathbb{N}_0$, because all real components of $m\rho$ have the same dimension r. So for the determinant of C, we have

$$\begin{aligned} 1 = |\det(C)| = \prod_{i=1}^{l} |\det(C_i)| &= \prod_{i=1}^{l} |\lambda_i|^{k_i r} \\ &= \left(\prod_{i=1}^{l} |\lambda_i|^{k_i} \right)^r \end{aligned}$$

and therefore $\prod_{i=1}^{l} |\lambda_i|^{k_i} = 1$. This means that C is not km hyperbolic, since $\sum_{i=1}^{l} k_i = km$ by looking at the size of C. $\square$

Proposition 10.16 finishes the proof of this direction of Theorem 10.3. Indeed, assume there exists a c-hyperbolic, integer-like matrix that commutes with $m\rho$, then the k of Proposition 10.16 must be strictly larger than $\frac{c}{m}$. This k is exactly the number of $\mathbb{R}$-irreducible components of ρ and thus we get that ρ has to split in strictly more than $\frac{c}{m}$ components over $\mathbb{R}$.

Proof of $\Uparrow$:

First we still need the following observation, which helps us construct matrices that commute with a given representation.

Lemma 10.17. *Let $M, C_1, \ldots, C_m$ be $k \times k$ matrices over any field K and assume that the matrices C_i commute with M for all i. Then the $km \times km$ matrices*

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & C_1 \\ \mathbb{1}_k & 0 & \ldots & 0 & C_2 \\ 0 & \mathbb{1}_k & \ldots & 0 & C_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbb{1}_k & C_m \end{pmatrix}, \quad B = \begin{pmatrix} M & 0 & \ldots & 0 \\ 0 & M & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M \end{pmatrix}$$

also commute.

Now let ρ be a $\mathbb{Q}$-irreducible representation that splits in more than $\frac{c}{m}$ components over $\mathbb{R}$. We will construct a c-hyperbolic, integer-like matrix C with coefficients in $\mathbb{Q}$ that commutes with $m\rho = \rho + \ldots + \rho$.

Let χ be the character of an irreducible component of ρ and take a field extension $\mathbb{Q} \subseteq K$ of minimal degree n such that χ is afforded by a K-representation $\rho_0 : G \to \mathrm{GL}(k, K)$. Note that n is also the number of components of ρ over $\mathbb{C}$.

First we construct a c-hyperbolic unit μ in some field extension of K. Fix a field extension $L \supseteq K$ of degree m, which is possible because of Theorem 4.21.

First assume that χ is not affordable by a real representation, so K, and hence also L, is totally imaginary. Since n is also the number of components of ρ over $\mathbb{C}$, we find that $n > 2\frac{c}{m}$ and thus $[L : \mathbb{Q}] > 2c$ ($\Rightarrow [L : \mathbb{Q}] \geq 2c + 2$). It follows from Proposition 4.14 that there exits a c-hyperbolic unit μ in E.

In the other case, χ is afforded by a real representation. Because of Theorem 10.13, we can assume that K is real and we can take L to be not totally imaginary. This time we have that $mn > c$ and thus there exists a c-hyperbolic unit μ in E as well. So in both cases we find a Galois extension E of F of degree m and a c-hyperbolic unit in E.

Let $\sigma_1, \ldots, \sigma_n$ be the n distinct monomorphisms of $K \to \mathbb{C}$ and take a matrix Q as in Theorem 10.8, with corresponding representation $\tilde{\rho} = Q^{-1}\rho Q$ just as in the proof. Look at the matrix

$$R = \mathbb{1}_m \otimes Q = \begin{pmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{pmatrix},$$

then it holds that

$$R(m\tilde{\rho})R^{-1} = \begin{pmatrix} Q\tilde{\rho}Q^{-1} & 0 & \cdots & 0 \\ 0 & Q\tilde{\rho}Q^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q\tilde{\rho}Q^{-1} \end{pmatrix} = m\rho.$$

Let

$$f_0(X) = \prod_{\sigma \in \mathrm{Gal}(L,K)} (X - \sigma(\mu)) = \sum_{j=0}^{m} b_j X^j \in F[X].$$

Take the polynomials $f_i = \sigma_i(f) = \sum_{j=0}^{m} a_{ij} X^j$ and form the matrices

$$C_j = \begin{pmatrix} a_{1j} & 0 & \dots & 0 \\ 0 & a_{2j} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nj} \end{pmatrix} \otimes \mathbb{1}_k = \begin{pmatrix} \sigma_1(b_j) & 0 & \dots & 0 \\ 0 & \sigma_2(b_j) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n(b_j) \end{pmatrix} \otimes \mathbb{1}_k.$$

It's easy to see that every matrix C_j commutes with the representation $\tilde{\rho}$ and that $QC_jQ^{-1} \in \mathrm{GL}(kn, \mathbb{Q})$ because of Proposition 10.9. Now construct the matrix

$$\tilde{C} = \begin{pmatrix} 0 & 0 & \dots & 0 & -C_0 \\ \mathbb{1}_{kn} & 0 & \dots & 0 & -C_1 \\ 0 & \mathbb{1}_{kn} & \dots & 0 & -C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbb{1}_{kn} & -C_{m-1} \end{pmatrix},$$

then it is obvious that $\tilde{C}$ commutes with $m\tilde{\rho}$ because of Lemma 10.17. A direct computation shows that

$$R\tilde{C}R^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & -QC_0Q^{-1} \\ QQ^{-1} & 0 & \dots & 0 & -QC_1Q^{-1} \\ 0 & QQ^{-1} & \dots & 0 & -QC_2Q^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & QQ^{-1} & -QC_{m-1}Q^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 & -QC_0Q^{-1} \\ \mathbb{1}_{kn} & 0 & \dots & 0 & -QC_1Q^{-1} \\ 0 & \mathbb{1}_{kn} & \dots & 0 & -QC_2Q^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbb{1}_{kn} & -QC_{m-1}Q^{-1} \end{pmatrix}$$

and thus $R\tilde{C}R^{-1} \in \mathrm{GL}(knm, \mathbb{Q})$. Note that the characteristic polynomial $f(X)$ of $\tilde{C}$ equals $\left(\prod_{i=1}^{n} f_i(X)\right)^k$. It's easy to check that the polynomial f has coefficients in $\mathbb{Q}$ and all of its roots are conjugates of μ. This means that f is some power of the minimal polynomial of μ. Therefore $\tilde{C}$ is c-hyperbolic and integer-like because of our choice of μ. Since C satisfies all conditions of the theorem, this completes the other direction of Theorem 10.3.

How to check the conditions of Theorem 10.1?

In this paragraph we discuss how to check whether a representation satisfies the second condition of Theorem 10.1 or equivalently how to check whether an infra-nilmanifold modeled on a free nilpotent Lie group has an Anosov diffeomorphism. Let $\rho : H \to \mathrm{GL}(n, \mathbb{Q})$ be a representation of a finite group H with character χ and assume that the character table of H is known, including information about the Schur indices of all the irreducible representations.

There is a standard inner product on the class functions, defined as

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|H|} \sum_{h \in H} \chi_1(h)\overline{\chi_2(h)}$$

for class functions χ_1, χ_2. Since the irreducible characters form an orthogonal basis for the class functions, it is easy to decompose every character χ into a sum

$$\chi = n_1\chi_1 + \ldots + n_k\chi_k$$

for irreducible characters $\chi_i \in \mathrm{Irr}(H)$, since $n_i = \langle \chi, \chi_i \rangle$. So the $\mathbb{C}$-irreducible components and their multiplicity of ρ are easy to compute from the character table.

Two characters χ_i and χ_j correspond to the same $\mathbb{Q}$-irreducible component of ρ if and only they are Galois conjugate, i.e. if $\chi_i^\sigma = \chi_j$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi_i), \mathbb{Q})$. Hence we can group the characters according to the $\mathbb{Q}$-irreducible components of ρ. In this way, we know which $\mathbb{Q}$-irreducible components occur in ρ and what their multiplicity is.

Fix a $\mathbb{Q}$-irreducible component of ρ, given by an irreducible character χ_i. Let k is the degree of the field extension and m the Schur Index of χ_i. To check whether there exists a c-hyperbolic automorphism commuting with ρ, we have to compare its multiplicity n_i with the number of components over $\mathbb{R}$. The number of real components is equal to km if χ_i is afforded by a real representation and otherwise the number of real components is $\frac{km}{2}$. To check whether χ_i is afforded by a real representations, we can compute the invariant $\nu_2(\chi_i)$ of this character and from Section 10.2.1 this shows us whether χ_i is afforded by a real representation.

So from the character table of H we can check the conditions of Theorem 10.1. An important ingredient is the Schur index of the $\mathbb{C}$-irreducible representations of H.

10.4 Some applications of Theorem 10.1

In [40] it was already shown how Theorem 9.14 could be used to construct infra-nilmanifolds with an abelian holonomy group and allowing an Anosov diffeomorphism. In this section, we will now show how we can also apply this theorem in the case of non-abelian holonomy groups. We first recall some of the facts which were developed in [40].

Let N be a torsion–free, finitely generated nilpotent group N, then we define for all positive integers i the subgroup

$$\Gamma_i(N) = \sqrt{\gamma_i(N)} = \{x \in N \mid \exists n \in \mathbb{N}_0 : x^n \in \gamma_i(N)\} = N \cap \gamma_i(N_{\mathbb{Q}}),$$

where the $\gamma_i(N)$ indicate the terms of the lower central series of N. These groups $\Gamma_i(N)$ are fully characteristic subgroups of N.

The following theorem which was proved in [40, Theorem 4.1] is very useful to find examples of infra-nilmanifolds with a specific rational holonomy representation.

Theorem 10.18. *Let $\varphi : H \to \mathrm{Aut}(N)$ be a faithful representation of a finite group H into the group of automorphisms of a torsion-free, finitely generated nilpotent group N, and denote with*

$$\bar{\varphi}_i : H \to \mathrm{Aut}\left({}^{\Gamma_i(N)}\!/_{\Gamma_{i+1}(N)}\right) \cong \mathrm{Aut}(\mathbb{Z}^{k_i})$$

the induced morphism. If there exists, for some positive integer i, a torsion-free extension

$$1 \to {}^{\Gamma_i(N)}\!/_{\Gamma_{i+1}(N)} \to \bar{\Gamma} \to H \to 1$$

inducing $\bar{\varphi}_i$, then there exists an almost–Bieberbach group Γ with holonomy group H, whose translation subgroup is a finite index subgroup of N and such that the rational holonomy representation $\psi: H \to \mathrm{Aut}(N^{\mathbb{Q}})$ coincides with $\varphi : H \to \mathrm{Aut}(N) < \mathrm{Aut}(N^{\mathbb{Q}})$.

Now, we let $N_{r,c}$ be the free c–step nilpotent group on r generators and we use $N_{r,c,\mathbb{Q}}$ to denote the rational Mal'cev completion of $N_{r,c}$. The corresponding Lie group (so the real Mal'cev completion) is the free c–step nilpotent Lie group on r generators. We have the natural homomorphisms

$$\mu : \mathrm{Aut}(N_{r,c}) \to \mathrm{Aut}\left({}^{N_{r,c}}\!/\!_{[N_{r,c}, N_{r,c}]} \right) \simeq \mathrm{GL}(r, \mathbb{Z}) \text{ and}$$

$$\mu_{\mathbb{Q}} : \mathrm{Aut}(N_{r,c,\mathbb{Q}}) \to \mathrm{Aut}\left({}^{N_{r,c,\mathbb{Q}}}\!/\!_{[N_{r,c,\mathbb{Q}}, N_{r,c,\mathbb{Q}}]} \right) \simeq \mathrm{GL}(r, \mathbb{Q}),$$

which are both onto. For making the identifications with $\mathrm{GL}(r, \mathbb{Z})$ and $\mathrm{GL}(r, \mathbb{Q})$ we fix a set of generators $x_1, x_2, \ldots, x_r$ of $N_{r,c}$ and the images of these these generators in the respective abelianizations give rise to a basis of the free abelian group ${}^{N_{r,c}}\!/\!_{[N_{r,c}, N_{r,c}]}$ and the $\mathbb{Q}$–vector space ${}^{N_{r,c,\mathbb{Q}}}\!/\!_{[N_{r,c,\mathbb{Q}}, N_{r,c,\mathbb{Q}}]}$ w.r.t. which we represent an automorphism as a matrix.
We will use the following observation:

Lemma 10.19. *Let $F < \mathrm{GL}(r, \mathbb{Z})$ be a finite subgroup. Then there exist a finite subgroup $\tilde{F}$ of $\mathrm{Aut}(N_{r,c,\mathbb{Q}})$ and a finitely generated subgroup $N \leq N_{r,c,\mathbb{Q}}$ such that*

(1) $\mu_{\mathbb{Q}}(\tilde{F}) = F$,

(2) *N contains $N_{r,c}$ as a subgroup of finite index,*

(3) $N\gamma_2(N_{r,c,\mathbb{Q}}) = N_{r,c}\gamma_2(N_{r,c,\mathbb{Q}})$ *and*

(4) $\forall \alpha \in \tilde{F} : \ \alpha(N) = N$.

Proof. The existence of a finite subgroup $\tilde{F} \leq \mathrm{Aut}(N_{r,c,\mathbb{Q}})$ such that $\mu_{\mathbb{Q}}(\tilde{F}) = F$ is a result which is due to Kuz'min (see [71, page 91]). Now, consider $N_{r,c,\mathbb{Q}} \rtimes \tilde{F}$ and let $\tilde{N}$ be the subgroup of $N_{r,c,\mathbb{Q}} \rtimes \tilde{F}$ which is generated by $N_{r,c}$ and $\tilde{F}$. As both $N_{r,c}$ and $\tilde{F}$ are finitely generated, we have that $\tilde{N}$ is finitely generated and hence also $N = \tilde{N} \cap N_{r,c,\mathbb{Q}}$, which is of finite index in $\tilde{N}$, is finitely generated. By this construction N satisfies properties (2), (3) and (4) in the statement of this lemma. □

We can now prove the following:

Theorem 10.20. *Let H be any finite group and c be any positive integer. Then there exists a positive integer K such that for any $k \geq K$ there is a infra-nilmanifold which is modeled on the free c–step nilpotent Lie group on k generators, admits an Anosov diffeomorphism and has H as its holonomy group.*

Proof. It is well known that any finite group H can be realized as the holonomy group of a flat manifold ([4]). Hence, there exists a representation

$$\psi : H \to \mathrm{GL}(n, \mathbb{Z})$$

and a torsion-free extension

$$0 \to \mathbb{Z}^n \to \bar{\Gamma}_1 \to H \to 1 \tag{10.1}$$

inducing ψ. Now, take $K = (c+1)(n+1)$ and choose any $k \geq K$. Let $\varphi_1 : H \to \mathrm{GL}(k, \mathbb{Z})$ be the representation

$$\varphi_1 = \underbrace{\psi \oplus \psi \oplus \cdots \oplus \psi}_{c+1 \text{ times}} \oplus \underbrace{1 \oplus 1 \oplus \cdots \oplus 1}_{k-(c+1)n \text{ times}}$$

where 1 denotes the trivial 1–dimensional representation. Note that there are at least $c+1$ of these trivial factors. Let f be a 2-cocycle such that the cohomology class $\langle f \rangle \in H^2(H, \mathbb{Z}^n)$ describes the extension (10.1), where of course $\mathbb{Z}^n$ is a H–module via ψ. We can decompose the H–module $\mathbb{Z}^k$ (via φ_1) as a direct sum $\mathbb{Z}^n \oplus \mathbb{Z}^{k-n}$ where H acts on the $\mathbb{Z}^n$–part via ψ and on the $\mathbb{Z}^{k-n}$–part via c times ψ and $k-(c+1)n$ times the trivial representation. Then $H^2(H, \mathbb{Z}^k) = H^2(H, \mathbb{Z}^n) \oplus H^2(H, \mathbb{Z}^{k-n})$ and the 2–cohomology class corresponding to $\langle f \rangle \oplus \langle 0 \rangle$ determines an extension

$$0 \to \mathbb{Z}^k \to \bar{\Gamma} \to H \to 1$$

which is also torsion-free and which induces $\varphi_1 : H \to \mathrm{GL}(k, \mathbb{Z})$.
Now, let $N_{k,c}$ be the free c–step nilpotent group on k generators. By Lemma 10.19, we can find a group N containing $N_{k,c}$ as a subgroup of finite index and a morphism $\varphi : H \to \mathrm{Aut}(N)$ which induces φ_1 on

$$\begin{aligned} {}^{\Gamma_1(N)}\!/\!_{\Gamma_2(N)} &= {}^{N}\!/\!_{N \cap \gamma_2(N_{k,c,\mathbb{Q}})} \\ &= {}^{N\gamma_2(N_{k,c,\mathbb{Q}})}\!/\!_{\gamma_2(N_{k,c,\mathbb{Q}})} \\ &= {}^{\Gamma_1(N_{k,c})}\!/\!_{\Gamma_2(N_{k,c})} \\ &\simeq \mathbb{Z}^k. \end{aligned}$$

By applying Theorem 10.18 we can conclude that there exists an almost–Bieberbach group Γ with holonomy group H and whose translation subgroup is of finite index in N. Hence, Γ determines an infra-nilmanifold M which is modeled on the free c–step nilpotent Lie group on k generators and has H as its holonomy group. Moreover, the rational holonomy representation of M coincides with $\varphi : H \to \mathrm{Aut}(N) \leq \mathrm{Aut}(N_{k,c,\mathbb{Q}})$ and it is obvious that the abelianized holonomy representation, then coincides with $\varphi_1 : H \to \mathrm{GL}(k, \mathbb{Z}) \leq \mathrm{GL}(k, \mathbb{Q})$. By construction, each $\mathbb{Q}$–irreducible component of φ_1 occurs with at least multiplicity $c+1$, and hence the conditions of Theorem 9.14 are trivially satisfied, which allows us to conclude that M admits an Anosov diffeomorphism $\square$

The previous result shows that any finite group can appear as the holonomy group of an infra-nilmanifold with an Anosov diffeomorphism and which is modeled on a free c–step nilpotent Lie group, but we have to allow large enough dimensions. For the next result, we will fix the smallest non-abelian group and determine such an infra-nilmanifold of minimal dimension.

So let $H = D_3 = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1\rangle$ be the dihedral group of order six (or the symmetric group of degree 3). The conjugacy classes of D_3 are $\{1\}$, $\{a, a^2\}$ and $\{b, ab, a^2b\}$. The character table of D_3 is given by

	1	a	b
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

.

These characters correspond to the representations

$$\rho_1 : D_3 \to \mathrm{GL}(1, \mathbb{Q}) \text{ with } \rho_1(a) = 1 \text{ and } \rho_1(b) = 1;$$

$$\rho_2 : D_3 \to \mathrm{GL}(1, \mathbb{Q}) \text{ with } \rho_1(a) = 1 \text{ and } \rho_1(b) = -1;$$

$$\rho_3 : D_3 \to \mathrm{GL}(2, \mathbb{Q}) \text{ with } \rho_1(a) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \rho_1(b) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

It follows that the $\mathbb{Q}$–irreducible representations of H are also $\mathbb{C}$–irreducible. Note that all representations are already given as integral representations. We remark that the representation

$$\rho_3' : H \to \mathrm{GL}(2, \mathbb{Q}) \text{ with } \rho_1(a) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \rho_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is equivalent to ρ_3 when considered as a representation over $\mathbb{Q}$, but they are not when considered over $\mathbb{Z}$. Hence, there are 4 non-equivalent irreducible $\mathbb{Z}$–representations of H, see e.g. [16].

Now assume that $M = \Gamma\backslash G$ is an infra-nilmanifold which is modeled on a free c–step nilpotent Lie group, which has H as its holonomy group and such that M admits an Anosov diffeomorphism. The abelianized rational holonomy representation $\varphi : D_3 \to \mathrm{GL}(n, \mathbb{Q})$ of M must be faithful and hence it must contain at least one component which is $\mathbb{Q}$–equivalent to ρ_3. As this component is $\mathbb{R}$–irreducible and M admits an Anosov diffeomorphism, it must in fact appear at least $c+1$ times by Theorem 9.14. We will now show that this lower bound is sharp.

Proposition 10.21. *Let $c > 1$. Then there exists an infra-nilmanifold M which is modeled on a free c–step nilpotent Lie group and such that M has holonomy group D_3, admits an Anosov diffeomorphism and its abelianized holonomy representation is equivalent to* $\underbrace{\rho_3 \oplus \rho_3 \oplus \cdots \oplus \rho_3}_{c+1 \text{ times}}$.

Before we can give the proof of this proposition, we need to recall one more result from [40] on totally reducible integral representations. A representation $\rho : F \to \mathrm{GL}(n, \mathbb{Z})$ is said to be totally reducible if and only if $\mathbb{Z}^n$ splits as a direct sum of $\mathbb{Z}$–irreducible submodules. The following lemma can be found in [40, Lemma 4.3]:

Lemma 10.22. *Let N be a finitely generated torsion-free nilpotent group and $\varphi : F \to \mathrm{Aut}(N_{\mathbb{Q}})$ a morphism, with F a finite group, such that $\varphi(f)(N) = N$ for all $f \in F$. Then there exists a finitely generated subgroup N' of $N_{\mathbb{Q}}$ such that*

- *$\varphi(f)(N') = N'$ for all $f \in F$,*
- *N is a subgroup of N' of finite index, and*
- *for all positive integers i, the induced representation*

$$\bar{\varphi}_i : F \to \mathrm{Aut}\left(\Gamma_i(N')/\Gamma_{i+1}(N')\right) = \mathrm{GL}(n_i, \mathbb{Z})$$

 is totally reducible.

Proof of Proposition 10.21. Let $N_{2(c+1),c}$ be the free nilpotent group on $2(c+1)$ generators and of class c. Let us denote the generators of $N_{2(c+1),c}$ by $x_1, x_2, \ldots, x_{2(c+1)}$. By Lemma 10.19 there exists a finitely generated subgroup N of $N_{2(c+1),c,\mathbb{Q}}$ containing $N_{2(c+1),c}$ as a subgroup of finite index and a morphism $\varphi : D_3 \to \mathrm{Aut}(N)$ such that the induced morphism $\varphi_1 : D_3 \to \mathrm{Aut}(\Gamma_1(N)/\Gamma_2(N)) = \mathrm{GL}(2(c+1), \mathbb{Z})$ is given by

$$\varphi_1 = \underbrace{\rho_3 \oplus \rho_3 \oplus \cdots \oplus \cdots \oplus \rho_3}_{c+1 \text{ times}}.$$

Now let $y_{i,j} = [x_i, x_j]$, $(1 \leq i < j \leq 2(c+1))$. The natural projections $\bar{y}_{i,j}$ of these elements in ${}^{\Gamma_2(N)}\!/\!{}_{\Gamma_3(N)}$ form a basis of a free abelian group which is of finite index in this quotient. We claim that the subgroup, say L, generated by $\bar{y}_{1,3}, \bar{y}_{1,4}, \bar{y}_{2,3}, \bar{y}_{2,4}$ is a D_3 submodule for the action of D_3 given by $\varphi_2 : D_3 \to \operatorname{Aut}\left({}^{\Gamma_2(N)}\!/\!{}_{\Gamma_3(N)}\right)$. In fact, we can compute this action explicitly. Note that by the form of φ_1, we have that

$$\begin{array}{ll}
\varphi(a)(x_1) = x_2 \mod \Gamma_2(N) & \varphi(b)(x_1) = x_2^{-1} \mod \Gamma_2(N) \\
\varphi(a)(x_2) = x_1^{-1}x_2^{-1} \mod \Gamma_2(N) & \varphi(b)(x_2) = x_1^{-1} \mod \Gamma_2(N) \\
\varphi(a)(x_3) = x_4 \mod \Gamma_2(N) & \varphi(b)(x_3) = x_4^{-1} \mod \Gamma_2(N) \\
\varphi(a)(x_4) = x_3^{-1}x_4^{-1} \mod \Gamma_2(N) & \varphi(b)(x_4) = x_3^{-1} \mod \Gamma_2(N)
\end{array}$$

From these identities we can compute the action of a and b on each $y_{i,j}$ mod $\Gamma_3(N)$, e.g.

$$\begin{aligned}
\varphi(a)(y_{1,4}) &= [\varphi(a)(x_1), \varphi(a)(x_4)] \bmod \Gamma_3(N) \\
&= [x_2, x_3^{-1}x_4^{-1}] \bmod \Gamma_3(N) \\
&= [x_2, x_3]^{-1}[x_2, x_4]^{-1} \bmod \Gamma_3(N) \\
&= y_{2,3}^{-1}y_{2,4}^{-1} \bmod \Gamma_3(N).
\end{aligned}$$

When doing this for all generators, we find that w.r.t. the basis $\bar{y}_{1,3}, \bar{y}_{1,4}, \bar{y}_{2,3}, \bar{y}_{2,4}$ of L, the action of a and b are represented by matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The character χ of this representation satisfies $\chi(1) = 4$, $\chi(a) = 1$ and $\chi(b) = 0$ and so $\chi = \chi_1 + \chi_2 + \chi_3$, from which it follows that the representation determined by the matrices A and B is $\mathbb{Q}$–equivalent to $\rho_1 \oplus \rho_2 \oplus \rho_3$. Now, we use Lemma 10.22 to find a subgroup N' of $N_{2(c+1),c,\mathbb{Q}}$ which contains N as a finite index subgroup, such that φ can also be seen as a representation $\varphi : D_3 \to \operatorname{Aut}(N')$ and such that for each i

$$\varphi_i : D_3 \to \operatorname{Aut}\left({}^{\Gamma_i(N')}\!/\!{}_{\Gamma_{i+1}(N')}\right)$$

is totally reducible. By the above, we know that φ_2 must have a subrepresentation which is $\mathbb{Q}$–equivalent to $\rho_1 \oplus \rho_2 \oplus \rho_3$. Hence as a representation over $\mathbb{Z}$ it must contain a subrepresentation which is $\mathbb{Z}$-equivalent to $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$ or to $\rho' = \rho_1 \oplus \rho_2 \oplus \rho_3'$. Both of these representations occur as the holonomy representation of a 4-dimensional Bieberbach group ([16]). Hence, by an analogous argument as in the proof of Theorem 10.20, we can conclude that there is a torsion-free extension

$$1 \to {}^{\Gamma_2(N')}\!\diagup_{\Gamma_3(N')} \to \bar{\Gamma} \to D_3 \to 1$$

inducing the representation φ_2.

It follows from Theorem 10.18 that there exists an almost–Bieberbach group Γ with holonomy group D_3 and such that the corresponding rational holonomy representation $\psi : D_3 \to \operatorname{Aut}(N_{2(c+1),c,\mathbb{Q}})$ coincides with φ. This implies that the abelianized rational holonomy representation is $\mathbb{Q}$–equivalent to $(c+1)\rho_3$, hence by our main Theorem 9.14, the infra-nilmanifold determined by Γ admits an Anosov diffeomorphism, which finishes the proof. □

10.5 Generalization to other classes of infra-nilmanifolds

Up till now, we discussed infra–nilmanifolds modeled on a free c–step nilpotent Lie group, but in fact the results of this chapter do generalize quite immediately to other classes of infra–nilmanifolds.

There are two crucial step in the arguments showing that Theorem 10.2 about the abelianized holonomy representation holds. One part is that every k-fold product of eigenvalues $\lambda_{i_1} \ldots \lambda_{i_k}$ for $k \leq c$ appears as an eigenvalue of the map on the Lie group G (except for possibly the squares of λ_i). The second step is that every element of $\mathrm{GL}(n, \mathbb{Q})$ induces some automorphism on the Lie group G. There are other Lie groups satisfying these two properties and the main result of this chapter also works on infra-nilmanifolds modeled on these Lie groups.

For example we can consider the Lie algebra $\mathfrak{g}_{c,d,r}$ which is the free c-step nilpotent and d–step solvable Lie algebra on r generators (over $\mathbb{R}$) and let $G_{c,d,r}$ be the corresponding simply connected Lie group. Theorem B. of [38] (which was slightly reformulated in [40, Theorem 2.1]) can now also be stated for manifolds modeled on $G_{c,d,r}$.

Theorem 10.23. *Let M be an infra-nilmanifold modeled on $G_{c,d,r}$, with holonomy group H and associated abelianized rational holonomy representation $\overline{\varphi} : H \to \mathrm{GL}(r, \mathbb{Q})$. Then the following are equivalent:*

M admits an Anosov diffeomorphism.

$$\Updownarrow$$

There exists an integer-like c-hyperbolic matrix $C \in \mathrm{GL}(r, \mathbb{Q})$ that commutes with every element of $\overline{\varphi}(H)$.

As explained above, the proof of this theorem follows almost word by word the original proof in [38].

Having obtained this theorem, we now also get the following generalization of our main result for free.

Theorem 10.24. *Let M be an infra-nilmanifold modeled on $G_{c,d,r}$, with holonomy group H and associated abelianized rational holonomy representation $\overline{\varphi} : H \to \mathrm{GL}(r, \mathbb{Q})$. Then the following are equivalent:*

M admits an Anosov diffeomorphism.

$$\Updownarrow$$

Every $\mathbb{Q}$-irreducible component of $\bar{\varphi}$ that occurs with multiplicity m, splits in more than $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

Chapter 11

Concluding remarks and open questions

As is often the case in research, the results of this dissertation open directions for further research and raise new questions. Some of these problems were already mentioned in the previous chapters, but this chapter gives a summary and adds some new results to build connections with other recent results. In some cases, we give ideas on how to tackle these problems.

11.1 Generalizations of expanding maps

By Theorem 3.21 the only closed manifolds, up to homeomorphism, on which expanding maps can be studied are the infra-nilmanifolds. In this section, we attempt to generalize the notion of expanding maps to the group level. The naive generalization to expanding group morphisms only exists on virtually nilpotent groups and this motivates why we should consider a different generalization. We also explain how NIL-affine crystallographic actions fit in the picture if one wants to study these properties on virtually polycyclic groups.

Expanding group morphisms

Expanding maps can be defined on any metric space, not only on closed Riemannian manifolds. For example, the positively expansive maps of Section

3.5 where defined on general compact metric spaces. In particular, expanding maps can also be defined on finitely generated groups equipped with the word metric.

Definition 11.1. Let G be a finitely generated group with finite generating set S. A group morphism $\varphi : G \to G$ is called **expanding** if there exists constants $c > 0$ and $\lambda > 1$ such that

$$d_S(\varphi^n(g), \varphi^n(h)) \geq c\lambda^n d_S(g, h)$$

for all $g, h \in G$.

It is a standard argument to show that this definition does not depend on the choice of generating set S for the group G by changing the constant c in the definition.

We give some examples of expanding group morphisms.

Example 11.2. The unique group morphism on the trivial group $G = \{e\}$ is expanding. In Lemma 11.6 we will show that this is the only finite group having an expanding group morphism.

Example 11.3. Let $f : M \to M$ be an expanding map on a closed Riemannian manifold M. The induced group morphism $f_* : \pi_1(M) \to \pi_1(M)$ on the fundamental group is an expanding group morphism. It is a consequence of Theorem 2.18 that the only expanding group morphisms on almost-Bieberbach groups Γ are constructed in this way.

Example 11.4. Consider the free product $G = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ with generators a, b and c for the first, second and third factor respectively. Let $\varphi : G \to G$ be the group morphism given by

$$\varphi(a) = bab$$

$$\varphi(b) = cbc$$

$$\varphi(c) = aca$$

on the generators a, b and c. Take $S = \{a, b, c\}$ as generating set for G. Then $d_S(\varphi(g), \varphi(h)) = 3d_S(g, h)$ and thus φ is an expanding group morphism. Note that $\varphi(G)$ is a subgroup of infinite index in G, since the subgroup generated by $abc \in G$ has trivial intersection with $\varphi(G)$.

The following lemmas state the first properties of expanding group morphisms.

Lemma 11.5. *Every expanding group morphism is injective.*

Proof. Let $\varphi : G \to G$ be an expanding group morphism with constants c and λ as in Definition 11.1. Assume that $\varphi(g) = e$ for some $g \in G$, then

$$0 = d_S(e, e) = d_S(\varphi(g), e) \geq c\lambda d_S(g, e) \geq 0$$

and thus $d_S(g, e) = 0$. □

Lemma 11.6. *The only finite group admitting an expanding group morphism is the trivial group.*

Proof. In Example 11.2 we showed that the trivial group has an expanding group morphism. Let G be a finite group with any generating set S and $\varphi : G \to G$ an expanding group morphism with parameters c and λ as in Definition 11.1. Let D be the diameter of the group G under the word metric d_S.

Assume there exists $g \in G$ with $g \neq e$, so $d_S(g, e) \geq 1$, and take $n \in \mathbb{N}$ such that $c\lambda^n d_S(g, e) > D$. This implies that

$$d_S(\varphi^n(g), e) \geq c\lambda^n d_S(g, e) > D$$

which is a contradiction. We conclude that G must be the trivial group. □

Corollary 11.7. Let $\varphi : G \to G$ be an expanding group morphism on a nilpotent group G, then G is torsion-free.

Proof. Let T be the torsion subgroup of G, then $\varphi(T) \leq T$ since T is fully characteristic. Thus φ induces a group morphism on the subgroup T and one can check that this restriction is also expanding. Since T is a finite group, Lemma 11.6 implies that T is trivial or thus that G is torsion-free. □

The following example shows that Corollary 11.7 does not hold for virtually abelian groups and so certainly not for virtually nilpotent groups.

Example 11.8. Let G be the infinite dihedral group, given by the presentation

$$G = \langle a, b \mid b^2 = e, bab = a^{-1} \rangle.$$

The group G is virtually abelian, since the group generated by a forms a subgroup of index 2, but G is not torsion-free because $b^2 = e$. Let $\varphi : G \to G$ be the group morphism given by $\varphi(a) = a^2$ and $\varphi(b) = ab$ on the generators a and b. Take the finite generating set $S = \{a, b\}$. The morphism φ satisfies

$$d_S(\varphi(g), \varphi(h)) = 2d_S(g, h)$$

for all $g, h \in G$ and thus φ is an expanding group morphism.

Example 11.8 shows that the infinite dihedral group does admit an expanding group morphism. On the other hand, the group $\mathbb{Z} \oplus \mathbb{Z}_2$ does not admit an expanding group morphism since the torsion elements form a non-trivial finite subgroup which is fully characteristic, see the proof of Corollary 11.7. So the existence of an expanding group morphism on a virtually abelian group is not invariant under commensurability, contrary to the situation for almost-Bieberbach groups.

Theorem 11.16 classifies the almost-Bieberbach groups admitting an expanding group morphism. The question which virtually nilpotent groups admit an expanding group morphism is still open.

Question 11.1. Which virtually nilpotent groups admit an expanding group morphism?

Note that the proof of Main Theorem 1 works for almost-crystallographic groups Γ but not every expanding affine infra-nilmanifold endomorphism $\bar{\alpha}$ induces an expanding group morphism on Γ. So the question is open even in the case of almost-crystallographic groups.

As a consequence of the polynomial growth theorem of Gromov we show that expanding group morphisms such that the image is of finite index in the whole group exist only on virtually nilpotent groups.

Theorem 11.9. *Let G be a group and assume that $\varphi : G \to G$ is an expanding group morphism. If $\varphi(G)$ has finite index in G then the group G is virtually nilpotent.*

This theorem is the equivalent of Theorem 3.21 for the case of groups.

Proof. By Theorem 3.22, it suffices to show that G has polynomial growth. By taking a power of φ, we can assume that

$$d_S(\varphi(g), \varphi(h)) > 2d_S(g, h)$$

for all $g, h \in G$. Since $\varphi(G)$ is of finite index in G, there exists some constant $c > 0$ such that for every $g \in G$, there exists an $h \in G$ with $d_S(g, \varphi(h)) \leq c$. Denote by d the number of elements in the ball of radius c. We write $|X|$ for the number of elements in a subset $X \subseteq G$.

Now take $r > 0$ and consider the ball of radius r around the identity e, denoted as $B(r)$. We will give an estimate for $|B(r)|$ in terms of the number of elements in balls of smaller radius. Every element of $B(r)$ lies at distance at most c from an element in $\varphi(G)$, so

$$|B(r)| \leq d\,|B(r + c) \cap \varphi(G)|.$$

Because of the properties of φ we know that if $\varphi(g) \in B(r+c)$, then $g \in B\left(\frac{r+c}{2}\right)$. This implies that

$$|B(r)| \leq d\, |B\left(\frac{r+c}{2}\right)| = d\, |B\left(\frac{r}{2}+\frac{c}{2}\right)|.$$

By applying induction to the previous inequality, we conclude that

$$|B(r)| \leq d^n\, |B\left(\frac{r}{2^n}+\frac{c}{2}+\frac{c}{4}+\ldots+\frac{c}{2^n}\right)| \leq d^n\, |B\left(\frac{r}{2^n}+c\right)|$$

for every $n \in \mathbb{N}$ and $r > 0$.

Now fix $l \in \mathbb{N}$ such that $d \leq 2^l$ and take $r > 0$ arbitrary. Let $k \in \mathbb{N}$ be the integer such that $2^k \leq r \leq 2^{k+1}$. By applying the previous inequality for $n = k+1$ we get

$$|B(r)| \leq d^{k+1}\, |B(1+c)| \leq 2^{l(k+1)}\, |B(1+c)| \leq 2^l\, |B(1+c)| r^l.$$

We conclude that G has polynomial growth. □

So the natural generalization of expanding maps exists only on virtually nilpotent groups, under the mild condition that the image is of finite index. For virtually polycyclic groups, this condition is even always satisfied.

Definition 11.10. A group G is called **polycyclic** if it has a finite series of subgroups

$$\{e\} = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

such that G_i is normal in G_{i+1} and the quotient ${}^{G_{i+1}}\!/_{G_i}$ is cyclic. The number of infinite cyclic groups ${}^{G_{i+1}}\!/_{G_i}$ is called the **rank** of G and this rank does not depend on the choice of finite series G_i.

Every finitely generated nilpotent group is polycyclic and thus polycyclic groups form a generalization of nilpotent groups. The rank of a virtually polycyclic group is defined as the rank of a polycyclic group of finite index, which is independent of the choice of this subgroup. Let G be a virtually polycyclic group and $\varphi : G \to G$ an injective group morphism. The group $\varphi(G)$ is isomorphic to G and thus has the same rank as G. This implies that the subgroup $\varphi(G)$ has finite index in G.

If a virtually polycyclic group admits an expanding group morphism, it follows from Lemma 11.5 and Theorem 11.9 that the group must be virtually nilpotent. So the natural generalization of expanding maps on virtually polycyclic groups only makes sense on virtually nilpotent groups. Therefore we look for another property characterizing expanding maps on almost-Bieberbach groups.

Dis-cohopfian groups

Every expanding group morphisms has the following property.

Theorem 11.11. *Let $\varphi : G \to G$ be an expanding group morphism of a group G, then*

$$\bigcap_{n\in\mathbb{N}} \varphi^n(G) = 1.$$

Proof. Take $g \in \bigcap_{n\in\mathbb{N}} \varphi^n(G)$ and assume that $g \neq 1$, thus $d_S(g,e) > 0$. Let c, λ be the parameters of the expanding map φ as in Definition 11.1 and take n such that $c\lambda^n > d_S(g,e)$. By the definition of g, there exists $h \in G$ such that $\varphi^n(h) = g$. Since φ is injective, $h \neq e$ and thus $d_S(h,e) \geq 1$. The computation

$$d_S(g,e) = d_S(\varphi^n(h), \varphi^n(e)) > c\lambda^n d_S(h,e) \geq c\lambda^n > d_S(g,e)$$

gives us a contradiction and we conclude that $\bigcap_{n\in\mathbb{N}} \varphi^n(G)$ only contains the trivial element. $\square$

Note that there are injective group morphisms $\varphi : \Gamma \to \Gamma$, even on almost-Bieberbach groups Γ, which are not expanding but which do satisfy

$$\bigcap_{n\in\mathbb{N}} \varphi^n(G) = 1.$$

But on almost-Bieberbach groups the existence of such group a morphism does imply the existence of an expanding group morphism.

Theorem 11.12. *Let $\varphi : N \to N$ be an injective group morphism of an $\mathcal{F}$-group N such that*

$$\bigcap_{n\in\mathbb{N}} \varphi^n(N) = 1.$$

Every invariant factor $p_i(X) \in \mathbb{Z}[X]$ of $\varphi^{\mathbb{Q}} \in \operatorname{Aut}(N^{\mathbb{Q}})$ satisfies $|p_i(0)| > 1$, i.e. the constant terms of the invariant factors are $\neq \pm 1$.

Proof. For simplifying notations, we denote by $\varphi^{\mathbb{Q}}$ also the automorphism induced by φ on the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. Assume that $\varphi^{\mathbb{Q}}$ has an invariant factor $p_i(X)$ with $p_i(0) = \pm 1$, then we will show that

$$\bigcap_{n\in\mathbb{N}} \varphi^n(N) \neq 1.$$

From Theorem 5.7 it follows that there exists a subspace $V^{\mathbb{Q}} \subseteq \mathfrak{n}^{\mathbb{Q}}$ such that $\varphi^{\mathbb{Q}}(V^{\mathbb{Q}}) = V^{\mathbb{Q}}$ and $p_i(X)$ is the characteristic polynomial of the restriction

$\varphi^{\mathbb{Q}}\big|_{V^{\mathbb{Q}}}$. Let $v_1, \ldots, v_k$ be the basis vectors of a basis for the subspace $V^{\mathbb{Q}}$ such that the matrix representation of $\varphi^{\mathbb{Q}}\big|_{V^{\mathbb{Q}}}$ is $L(p_i)$. Consider the group elements $\exp(v_1)$, $\ldots$, $\exp(v_n) \in N^{\mathbb{Q}}$. There exists some integer $n > 0$ such that $\exp(nv_i) \in N$ for every i, so we can assume without loss of generality that $\exp(v_i) \in N$ for every i.

Take the subgroup H generated by the elements $\exp(v_i)$ and consider the lattice group H^{lat} containing H. Similarly as in Proposition 6.2 (since the matrix representation of $\varphi^{\mathbb{Q}}$ has integer entries in the basis v_i), it follows that $\varphi(H^{\text{lat}}) \leq H^{\text{lat}}$. The restriction of φ to $V^{\mathbb{Q}}$ and therefore also to $\left(H^{\text{lat}}\right)^{\mathbb{Q}}$ only has eigenvalues which are algebraic integers by construction. Therefore the determinant of φ restricted to $\left(H^{\text{lat}}\right)^{\mathbb{Q}}$ is ± 1, see Proposition 4.5, and hence it holds that $\varphi(H^{\text{lat}}) = H^{\text{lat}}$. This implies that $1 \neq H \leq \bigcap_{n \in \mathbb{N}} \varphi^n(N)$, which ends the proof. $\square$

It seems likely that the converse of Theorem 11.12 is true as well, namely that every injective group morphism satisfying the condition on their invariant factors satisfies $\cap_{n \in \mathbb{N}} \varphi^n(N) = 1$.

Conjecture 11.13. Let $\varphi : N \to N$ be an injective group morphism of an $\mathcal{F}$-group N such that the invariant factors of $\varphi^{\mathbb{Q}}$ have constant term > 1 in absolute value. Then the group morphism φ satisfies

$$\bigcap_{n \in \mathbb{N}} \varphi^n(N) = 1.$$

Theorem 11.12 has the following consequence by applying the techniques of Chapter 8.

Corollary 11.14. Let Γ be an almost-Bieberbach group. Then Γ has an injective group morphism $\varphi : \Gamma \to \Gamma$ such that

$$\bigcap_{n \in \mathbb{N}} \varphi^n(\Gamma) = 1$$

if and only if Γ has an expanding group morphism.

Proof. One implication is immediate from Theorem 11.11. For the other implication, let φ be an injective group morphism with

$$\bigcap_{n \in \mathbb{N}} \varphi^n(\Gamma) = 1.$$

Consider the restriction of φ to the Fitting subgroup N of Γ. By Theorem 11.12, $\varphi^{\mathbb{Q}}$ only has invariant factors $p_i(X) \in \mathbb{Z}[X]$ such that $|p_i(0)| > 1$. This implies that $N(\varphi^{\mathbb{Q}})$, as defined in Section 8.2, is expanding. In particular, the Lie group G has a positive grading and thus Γ has an expanding group morphism by Theorem 8.21. $\square$

For general groups, we study the group morphisms $\varphi : G \to G$ such that

$$\bigcap_{n \in \mathbb{N}} \varphi^n(G) = 1$$

since these exist also on virtually polycyclic groups which are not virtually nilpotent. Although these are not always expanding, their existence is equivalent to the existence of an expanding map by Corollary 11.14.

Definition 11.15. Let G be any group, then we call G **dis-cohopfian** if there exists an injective group morphism $\varphi : G \to G$ such that

$$\bigcap_{n \in \mathbb{N}} \varphi^n(G) = 1.$$

By combining Theorem 11.11 and Theorem 8.21, we get a full algebraic characterization of the dis-cohopfian almost-Bieberbach groups.

Theorem 11.16. *Let $\Gamma \leq \operatorname{Aff}(G)$ be an almost-Bieberbach group modeled on the Lie group G with corresponding Lie algebra $\mathfrak{g}$. Then the following statements are equivalent.*

1. *The infra-nilmanifold $\Gamma\backslash G$ admits an expanding map.*
2. *The group Γ is dis-cohopfian.*
3. *The Lie algebra $\mathfrak{g}$ has a positive grading.*

It is exactly this result that Y. Cornulier gave in [20] for nilpotent groups Γ. Note that expanding maps are never mentioned in this paper, the dis-cohopfian groups are studied for being a strong converse to cohopfian groups.

The following question is natural in the sense of Theorem 11.16 and asks whether a similar result is possible for virtually polycyclic groups.

Question 11.2. Is there an algebraic way to describe the virtually polycyclic groups which are (dis-)cohopfian?

A positive answer to this question also gives a way of tackling the following question of Y. Cornulier in [20].

Question 11.3. Is it true that the properties dis-cohopfian and cohopfian are invariant under taking finite index subgroups for virtually polycyclic groups?

By Theorem 11.16 and Theorem 8.22 this is true for the class of almost-Bieberbach groups.

NIL-affine crystallographic actions

For almost-Bieberbach groups, the postive answers on Question 11.2 and 11.3 follow from considering the group as a subgroup of

$$N^{\mathbb{Q}} \rtimes_\rho F$$

with $\rho : F \to \mathrm{Aut}(N^{\mathbb{Q}})$ the rational holonomy representation of the almost-Bieberbach group.

Therefore a good start for the case of polycyclic groups is to work in the context of NIL-affine crystallographic actions of these groups. A NIL-affine crystallographic action of a group Γ is a representation into the affine group $\mathrm{Aff}(N)$ of a simply connected and connected nilpotent Lie group N

$$\rho : \Gamma \to \mathrm{Aff}(N)$$

such that Γ acts properly discontinuous and cocompactly on N. In [29] it is shown that every virtually polycyclic group admits such a representation and that these representations are unique up to polynomial conjugation (where polynomial is defined in terms of a Mal'cev basis for the Lie group N).

These NIL-affine crystallographic actions thus form a generalization of almost-crystallographic groups in the sense that every almost-crystallographic groups is given by a represention

$$\rho : \Gamma \to N \rtimes F \leq \mathrm{Aff}(N)$$

with F a finite subgroup of automorphisms of N. For almost-crystallographic groups, we have Theorem 2.21 stating that for every group morphism $\varphi : \Gamma \to \Gamma$ there exists an affine map $\alpha \in \mathrm{aff}(N)$ such that

$$\forall \gamma \in \Gamma : \varphi(\gamma)\alpha = \alpha\gamma.$$

The advantage of this result is that every group morphism is more or less described by a linear map between Lie algebras, which are easier to handle.

A first step in understanding groups morphisms of virtually polycyclic groups is to find a similar statement for NIL-affine crystallographic actions. We believe that the following question is the correct generalization.

Question 11.4. Let $\varphi : \Gamma_1 \to \Gamma_2$ be a group morphism between two NIL-affine crystallographic groups $\Gamma_i \leq \operatorname{Aff}(N_i)$. Does there exists a polynomial map $p : N_1 \to N_2$ such that

$$\varphi(\gamma) \circ p = p \circ \gamma$$

holds $\forall \gamma \in \Gamma_1$?

These polynomial maps are easier to study than group morphisms and a positive answer could reveal much of the structure of virtually polycyclic groups.

A positive answer to Question 11.4 (even only in the case of injective group morphisms φ) could help to answer which virtually polycyclic groups are cohopfian or dis-cohopfian and study if this property is invariant under taking finite index subgroups. But this question also has many other important applications, e.g. in topological fixed point theory. Theorem 2.21 has led to a general formula for the Lefschetz fixed point number and Nielsen fixed point number for self-maps on infra-nilmanifolds, see [77], and there is hope to generalize this result using a positive answer to the question.

11.2 Quasi-isometric invariants

Another research question following from this thesis is to study quasi-isometric invariants of virtually nilpotent groups. We give some background on this question, a more detailed introduction into geometric group theory can be found in [25].

Definition 11.17. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called a **quasi-isometry** if there exist constants $a, b > 0$ such that

1. (Quasi-isometric embedding) For every $x_1, x_2 \in X$, we have that
$$\frac{d_X(x_1, x_2)}{a} - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b;$$
2. (Quasi-surjective) For every $y \in Y$, there exists some $x \in X$ such that $d_Y(y, f(x)) \leq b$.

In geometric group theory, one takes finitely generated groups G as a metric space where the metric d_S on G is given by the distance in the Cayley graph for a finite generating set S. One important property is that for two different generating sets S and S', the metric spaces (G, d_S) and $(G, d_{S'})$ are quasi-isometric. The group properties one studies in geometric group theory are then the properties which are invariant under quasi-isometries.

Example 11.18. Let G be a finitely generated group with generating set S. The growth function $\beta_{G,S} : \mathbb{N} \to \mathbb{N}$ is defined as the number of elements in the ball $B(n)$ of radius n around e. We consider the equivalence relation of quasi-equivalence on functions $\mathbb{N} \to \mathbb{N}$ as given by Definition 4.11. The equivalence class of $\beta_{G,S}$ is a quasi-isometric invariant of the group G.

Other examples of quasi-isometric invariants include hyperbolicity and the asymptotic cone of the group G.

A basic question in geometric group theory is to study which groups are quasi-isometric. Except for the trivial examples, namely finite index subgroups and quotients by finite normal subgroups, the easiest examples follow from the Schwarz-Milnor Lemma.

Lemma 11.19 (Schwarz-Milnor)**.** *Let Γ be a group acting by isometries on a Riemannian manifold M and assume that the action is properly discontinuously and cocompact. Then the group Γ is finitely generated and for every generating set S and point $m \in M$, the map*

$$\Gamma \to M$$

$$\gamma \mapsto {}^{\gamma}m$$

is a quasi-isometry.

In particular, if two groups act properly discontinuously and cocompactly on the same manifold M, these groups are quasi-isometric. If we apply this result to the class of almost-crystallographic groups, we get that any two almost-crystallographic groups modeled on the same nilpotent Lie group are quasi-isometric.

It is conjectured that the converse is also true, i.e. if two almost-crystallographic groups are quasi-isometric, then they are modeled on the same Lie group G.

Question 11.5. Let $\Gamma_1 \leq \operatorname{Aff}(G)$ and $\Gamma_2 \leq \operatorname{Aff}(H)$ be two almost-crystallographic groups which are quasi-isometric. Are the Lie groups G and H isomorphic, or equivalently are the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ associated to G and H isomorphic?

To tackle this problem, one has to study quas-isometric invariants of almost-crystallographic groups and to find an invariant which is distinct for groups modeled on different Lie groups . The first possibility is to consider the growth of these groups.

Example 11.20. A variant of the Schwarz-Milnor Lemma states that the growth function of a lattice is quasi-equivalent to the volume growth of the Lie group. The volume growth of $\mathbb{R}^n$ is quasi-equivalent to the map $\beta_0 : \mathbb{N} \to \mathbb{N}$ with $\beta_0(k) = k^n$, implying that the growth function $\beta_{\mathbb{Z}^n} \sim \beta_0$.

This formula can be generalized to arbitrary nilpotent groups. Let N be a nilpotent group of nilpotency class c and denote by d_i the rank of the abelian group ${}^{\gamma_i(N)}\!/\!_{\gamma_{i+1}(N)}$. Define $d = \sum_{i=1}^{c} i d_i$, then $\beta_{N,S}$ is quasi-equivalent to a polynomial of degree d for every generating set S. A proof can be found in [6].

Let $\mathfrak{g}$ be a Lie algebra with lower central series $\gamma_i(\mathfrak{g})$. The graded Lie algebra $\mathfrak{g}^{\mathrm{gr}}$ associated to $\mathfrak{g}$ is defined as the vector space

$$\mathfrak{g}^{\mathrm{gr}} = {}^{\gamma_1(\mathfrak{g})}\!/\!_{\gamma_2(\mathfrak{g})} \oplus {}^{\gamma_2(\mathfrak{g})}\!/\!_{\gamma_3(\mathfrak{g})} \oplus \cdots \oplus {}^{\gamma_c(\mathfrak{g})}\!/\!_{\gamma_{c+1}(\mathfrak{g})}$$

with the Lie bracket between the vectors $X + \gamma_{i+1}(\mathfrak{g}) \in {}^{\gamma_i(\mathfrak{g})}\!/\!_{\gamma_{i+1}(\mathfrak{g})}$ and $Y + \gamma_{j+1}(\mathfrak{g}) \in {}^{\gamma_j(\mathfrak{g})}\!/\!_{\gamma_{j+1}(\mathfrak{g})}$ given by

$$[X, Y] + \gamma_{i+j+1}(\mathfrak{g}) \in {}^{\gamma_{i+j}(\mathfrak{g})}\!/\!_{\gamma_{i+j+1}(\mathfrak{g})}.$$

The natural positive grading

$$\mathfrak{g}^{\mathrm{gr}} = \bigoplus_{i>0} \mathfrak{g}_i$$

on the Lie algebra $\mathfrak{g}^{\mathrm{gr}}$ is given by $\mathfrak{g}_i = {}^{\gamma_i(\mathfrak{g})}\!/\!_{\gamma_{i+1}(\mathfrak{g})}$. The first step for Question 11.5 was given by P. Pansu in [88].

Theorem 11.21. *Let $\Gamma_1 \leq \mathrm{Aff}(G)$ and $\Gamma_2 \leq \mathrm{Aff}(H)$ be two almost-crystallographic groups which are quasi-isometric. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to G and H, then the graded Lie algebras $\mathfrak{g}^{\mathrm{gr}}$ and $\mathfrak{h}^{\mathrm{gr}}$ are isomorphic.*

From Example 11.20, it follows that the growth function of an almost-crystallographic group $\Gamma \leq \mathrm{Aff}(G)$ depends only on the graded Lie algebra $\mathfrak{g}^{\mathrm{gr}}$. This shows that the growth function is not useful to further classify almost-crystallographic groups under quasi-isometries in the view of Theorem 11.21.

Therefore we need other quasi-isometric invariants of nilpotent groups. A more general result due to Y. Shalom in [95, Theorem 1.2.] states that the Betti numbers of quasi-isometric nilpotent groups are equal. From this result the author constructs an example showing that the converse of Theorem 11.21 is not true.

An interesting question in view of this thesis is if the existence of expanding maps is a quasi-isometric invariant.

Question 11.6. Let Γ_1 and Γ_2 be almost-Bieberbach groups which are quasi-isometric. Is it true that Γ_1 admits an expanding map if and only if Γ_2 admits an expanding map?

Since we showed that the existence of an expanding map only depends on the covering Lie group, this problem is a reasonable one to consider. A similar question can be asked about the existence of a non-trivial self-cover or equivalently whether being cohopfian is a quasi-isometric invariant. Note that admitting an Anosov diffeomorphism is not a quasi-isometric invariant of $\mathcal{F}$-groups, since there are nilmanifolds $\Gamma_1 \backslash G$ and $\Gamma_2 \backslash G$ modeled on the same Lie group G such that $\Gamma_1 \backslash G$ admits an Anosov diffeomorphisms but $\Gamma_2 \backslash G$ doesn't.

11.3 Periodic points

In Chapter 7 we gave a complete description of the eventually periodic points for affine infra-nilmanifold endomorphisms. In a similar way, we showed that the periodic points form a dense subset of the infra-nilmanifold. A complete description of the periodic points of such a map is still missing and would be very interesting.

Question 11.7. Is there a description of $\mathrm{Per}(\bar{\alpha})$ for affine infra-nilmanifold endomorphisms $\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G$?

By Theorem 7.20 and Corollary 7.21, this question is equivalent to describing the periodic points of nilmanifold endomorphisms which lie in the set $p(N^{\mathbb{Q}})$.

The examples in Section 7.4 showed the set $\mathrm{Per}(\bar{\alpha})$ is hard to compute even for concrete examples on the 2-torus $\mathbb{T}^2$. So a more doable problem is probably to study the set of eventually periodic points which are not periodic.

Question 11.8. For which affine infra-nilmanifold endomorphisms $\bar{\alpha}$ does it hold that the set

$$\mathrm{ePer}(\bar{\alpha}) \setminus \mathrm{Per}(\bar{\alpha})$$

is dense?

From Theorem 7.18 it follows that if $\bar{\alpha}$ has an eigenvalue which is a root of unity, then the condition of Question 11.8 is never fulfilled.

In Example 7.26 we gave several examples of maps on $\mathbb{T}^2$ which had this property. These examples seem to suggest that the following property completely describes the affine infra-nilmanifold endomorphisms $\bar{\alpha}$ with $\mathrm{ePer}(\bar{\alpha}) \setminus \mathrm{Per}(\bar{\alpha})$ dense.

Conjecture 11.22. Let $\bar{\alpha}$ be an affine infra-nilmanifold endomorphism induced by the affine transformation $\alpha = (g, \delta)$. The set $\mathrm{ePer}(\bar{\alpha}) \setminus \mathrm{Per}(\bar{\alpha})$ is dense if and only if every invariant factor $p(X) \in \mathbb{Z}[X]$ of δ satisfies $|p(0)| > 1$.

So combined with Conjecture 11.13, we believe that the maps with $\mathrm{ePer}(\bar{\alpha}) \setminus \mathrm{Per}(\bar{\alpha})$ dense in $\Gamma\backslash G$ are exactly the maps such that

$$\bigcap_{n\in\mathbb{N}} \alpha^n \Gamma \alpha^{-n} = 1.$$

Another problem arising from this thesis is to study the periodic points of maps induced by elements $\alpha \in \mathrm{aff}(G)$ on infra-nilmanifolds for which the linear part is not necessarily injective. In Section 7.5 it is shown how the reduction to maps induced by elements in $\mathrm{Endo}(G)$ on nilmanifolds works. It seems like most techniques of Chapter 7 can be generalized by restricting this group morphism to the simply connected and connected subgroup

$$G_0 = \bigcap_{n\in\mathbb{N}} \delta^n(G),$$

on which the group morphism δ will be injective. This shows again the relation to the previous section and the definition of dis-cohopfian groups.

11.4 Anosov diffeomorphisms on infra-nilmanifolds

There are still many open questions that remain about Anosov diffeomorphisms. We summarize the most important ones related to this thesis.

Anosov diffeomorphisms on nilmanifolds

In Chapter 9 we discussed a new method for constructing Anosov Lie algebras, which answered many open questions about these Lie algebras. There are some new questions which arise from these new examples.

In general, the construction of Chapter 9 only gives us existence of the Anosov Lie algebra. An explicit presentation of the Lie algebra, given by generators and brackets, is only possible in very specific cases. By refining these techniques it could be possible to describe these nilmanifolds in some specific cases and help us classify the Anosov Lie algebras of minimal type or admitting an Anosov automorphism of minimal signature. For example, in the case of nilpotency class 2, the Pfaffian form was a useful tool to study these Anosov Lie algebras. A general classification of the Anosov Lie algebras is not possible at this moment, but a classification of these subclasses may be.

In Section 9.3.3 we gave examples of Anosov automorphisms of minimal signature on Lie algebras of type $(2n, n, 2n, 2n, \ldots, 2n)$ and $(2n, n, 2n, n, 2n, \ldots)$ with $n > 1$. We conjecture that these types are the only possibilities.

Conjecture 11.23. Let $f : \mathfrak{n}^{\mathbb{Q}} \to \mathfrak{n}^{\mathbb{Q}}$ be an Anosov automorphism of minimal signature, then the type of $\mathfrak{n}^{\mathbb{Q}}$ is one of the following:

(i) $(2n, n, 2n, 2n, \ldots, 2n)$ or

(ii) $(2n, n, 2n, n, 2n, \ldots)$,

where $n > 1$.

To prove this conjecture, a careful study of unit Pisot numbers and their Galois conjugates is necessary. We gave a proof for Lie algebras of nilpotency class ≤ 4 in Chapter 9.

The first example of Anosov Lie algebras of type $(3, 3, \ldots, 3)$ was given in Section 9.3.4. We showed that there exist Lie algebras of types $(3, 3, 6)$ as well. It is still an open question to determine all possible types $(3, 3, n_3, \ldots, n_c)$ of Anosov Lie algebras. Also we don't know how many Anosov Lie algebra of type $(3, \ldots, 3)$ there are up to isomorphism.

The first example of a nilmanifold admitting an Anosov diffeomorphism but no expanding map was given in Section 9.3.5. The fundamental group of this example is of nilpotency class 6. It is unknown if this is an example of minimal nilpotency class.

Question 11.9. Do there exist nilmanifolds with fundamental group of nilpotency class < 6 which admit an Anosov diffeomorphism but no expanding map?

Again, this question requires a study of the Galois conjugates of a hyperbolic nilmanifold automorphism.

Anosov diffeomorphisms on infra-nilmanifolds

Another problem is to look for a complete description of infra-nilmanifolds admitting an Anosov diffeomorphism for other classes of infra-nilmanifolds than the ones modeled on free nilpotent Lie groups.

One possibility is to consider infra-nilmanifolds of dimension ≤ 6. The techniques of Chapter 9 also make it possible to construct finite groups of automorphisms on the Anosov Lie algebras.

Another possibility is to consider infra-nilmanifolds associated to graphs, as introduced in [22]. The question that S.G. Dani and M. Mainkar answer in [22] is which nilmanifolds $N_X \backslash G_X$ admit an Anosov diffeomorphism in terms of the graph X. In fact, they give a general expression for the connected component $\mathrm{Aut}(G_X)^o$ of the automorphism group of G_X. It is interesting to study what can be said in the case of infra-nilmanifolds.

Question 11.10. Can we give an algebraic description which infra-nilmanifolds modeled on Lie groups associated to graphs admit an Anosov diffeomorphism?

Because of Theorem 3.36 it suffices to study finite subgroups of $\mathrm{Aut}(G_X)$ and look which of them have a hyperbolic automorphism commuting with every element. Since finite subgroups typically do not lie in $\mathrm{Aut}(G_X)^o$, we need to extend the results of [22] to find information about $\mathrm{Aut}(G_X)$.

Note that in their paper [22], the authors say nothing about the nilmanifolds given by other lattices of G_X. The problem for general lattices of G_X seems very hard to solve, therefore we should start by considering infra-nilmanifolds with the group N_X as Fitting subgroup.

The Lie groups G_X associated to graphs X are always 2-step but possibly this idea can be generalized to construct infra-nilmanifolds modeled on Lie groups of higher nilpotency class. In fact, the Lie algebras $\mathfrak{g}_X$ are constructed from free 2-step nilpotent Lie algebras by dividing out the brackets given by the complement graph of X. If we start from a free Lie algebra of higher nilpotency class, we can still consider the ideal generated by these elements and divide out by this ideal.

Appendix

This appendix discusses some topics which were mentioned during the dissertation. Their details are discussed here to improve the readability of the thesis.

Lie groups associated to graphs

Nilmanifolds associated to graphs are introduced in the paper [22]. The main result of this paper gives a description of the automorphisms of the Lie group in terms of the graph. One of the consequences is a criterium to decide which nilmanifolds admit an Anosov diffeomorphism based on the properties of the graph. We discuss the details about these Lie groups here.

Let $X = (V, E)$ be a finite graph, where V is the set of vertices and E is the set of edges. We will assume that X is simple, i.e. that there are no loops, and undirected, meaning that the edges don't have an orientation. Every edge $e \in E$ is thus uniquely represented by its endpoints $v_1, v_2 \in V$ and we will write $e = v_1v_2$. Take U the vector space with basis given by the vertices V and W the vector space with basis E. We consider the Lie algebra $\mathfrak{g}_X = U \oplus W$ where the Lie bracket is given by the relations

$$[v_1, v_2] = v_1v_2$$

for all $v_1, v_2 \in V$ such that $v_1v_2 \in E$. The Lie algebra $\mathfrak{g}_X$ is nilpotent of nilpotency class ≤ 2 and has dimension $(\#V) + (\#E)$.

The Lie group G_X is the unique simply connected and connected nilpotent Lie group corresponding to the Lie algebra $\mathfrak{g}_X$. Because of the Baker-Campbell-Hausdorff formula, the Lie group G_X is given explicitly by the set $\mathfrak{g}_X$ where multiplication is given by

$$(u_1, w_1) * (u_2, w_2) = (u_1 + u_2, w_1 + w_2 + \frac{1}{2}[u_1, u_2]).$$

There is a natural lattice in these Lie groups, namely the discrete subgroup generated by the vertices $v \in V$. We will write this lattice as $N_X \leq G_X$.

Example 11.24. Take V any set of n vertices. For the complete graph (V, K), we get that G_X is the free 2-step nilpotent Lie group on n generators. The empty graph $(V, \varnothing)$ gives us the abelian Lie group $\mathbb{R}^n$.

The main result of [22] gives the connected component of the automorphism group $\mathrm{Aut}(G_X)$. We don't go into details about this result, since its formulation is rather technical.

Finite index subgroups

A crucial fact for proving Theorem 6.15 is that there are only a finite number of subgroups of a given index in a finitely generated group. In this section we give a proof of this well-known result. This result is also important for Example 9.7.

Theorem 11.1. *Let G be a finitely generated group and $n \in \mathbb{N}$ a natural number. Then there are only a finite number of subgroups $H \leq G$ such that $[G : H] = n$.*

First, let us recall the relation between finite index subgroups of G and actions of G on finite sets. Let $X = \{x_1, \ldots, x_n\}$ be a finite set and consider an action $G \curvearrowright X$ which is given by a group morphism $G \to S_n$ to the symmetric group. The set X can always be decomposed into its orbits under the action of G and thus there is no loss in generality by assuming that the action is transitive, i.e. that for every $x_i \in X$, there exists a $g \in G$ such that ${}^{g}x_1 = x_i$.

Consider the stabilizer H of the element $x_1 \in X$, so $H \leq G$ is the subgroup given by
$$H = \{g \in G \mid {}^{g}x_1 = x_1\}.$$
By fixing elements $g_i \in G$ such that ${}^{g_i}x_1 = x_i$, we find that G is the disjoint union of
$$G = \bigcup_{i=1}^{n} g_i H$$
of left cosets, where $g_i H$ is exactly the subset of G which maps x_1 to x_i. Therefore H is a subgroup of index n in G. The group H is uniquely determined by the action $G \curvearrowright X$ up to conjugation.

Vice versa, assume that $H \leq G$ is a subgroup such that $[G : H] = n$ and take X the set of left cosets
$$X = \{gH \mid g \in G\}.$$

Then G acts by left multiplication on the finite set X and the stabilizer of eH is equal to H.

The proof of Theorem 11.1 now follows by using the following easy lemma.

Lemma 11.2. *Let G be finitely generated and F a finite group, then there are only a finite number of group morphisms $G \to F$.*

Proof. A group morphism is completely determined by the images of the generators. Since there are only a finite number of possibilities for the images of generators, there are only a finite number of group morphisms. □

The lemma in combination with the discussion above implies Theorem 11.1.

Proof of Theorem 11.1. By the construction described above, every finite index subgroup H gives rise to an action $G \to S_n$ of a finite set $\{x_1, \dots, x_n\}$. This action uniquely determines the group H up to conjugation. Since every subgroup H of index n has only a finite number of conjugated subgroups and there are at most a finite number of actions of $G \to S_n$ by Lemma 11.2, the theorem holds. □

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List of publications

- **Periodic and eventually periodic points of affine infra-nilmanifold endomorphisms.** (Preprint)
- **Gradings on Lie algebras with applications to infra-nilmanifolds.** (Preprint, available at arXiv:1410.3713.)
- **Expanding maps and non-trivial self-covers on infra-nilmanifolds.** (joint with K. Dekimpe, available at arXiv:1407.8106.) To appear in Topological Methods in Nonlinear Analysis.
- **A new method for constructing Anosov Lie algebras.** (available at arXiv:1312.2872.) To appear in Transactions of the American Mathematical Society.
- **Anosov diffeomorphisms on infra-nilmanifolds modeled on free nilpotent Lie groups.** (joint with K. Dekimpe, available at arXiv:1304.6529.) To appear in Topological Methods in Nonlinear Analysis.

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